Hungary-Israel Binational 2007

Day 1

1 You have to organize a fair procedure to randomly select someone from n people so that every one of them would be chosen with the probability $\frac{1}{n}$. You are allowed to choose two real numbers $0 < p_1 < 1$ and $0 < p_2 < 1$ and order two coins which satisfy the following requirement: the probability of tossing "heads" on the first coin p_1 and the probability of tossing "heads" on the second coin is p_2. Before starting the procedure, you are supposed to announce an upper bound on the total number of times that the two coins are going to be flipped altogether. Describe a procedure that achieves this goal under the given conditions.

2 Let a, b, c, d be real numbers, such that $a^2 \le 1, a^2 + b^2 \le 5, a^2 + b^2 + c^2 \le 14, a^2 + b^2 + c^2 + d^2 \le 30$. Prove that $a + b + c + d \le 10$.

3 Let AB be the diameter of a given circle with radius 1 unit, and let P be a given point on AB. A line through P meets the circle at points C and D, so a convex quadrilateral $ABCD$ is formed. Find the maximum possible area of the quadrilateral.

Day 2

1 A given rectangle R is divided into mn small rectangles by straight lines parallel to its sides. (The distances between the parallel lines may not be equal.) What is the minimum number of appropriately selected rectangles areas that should be known in order to determine the area of R?

2 Given is an ellipse e in the plane. Find the locus of all points P in space such that the cone of apex P and directrix e is a right circular cone.

3 Let $t \ge 3$ be a given real number and assume that the polynomial $f(x)$ satisfies $|f(k) - t^k| < 1$, for $k = 0, 1, 2, \ldots, n$. Prove that the degree of $f(x)$ is at least n.