Cho a,b,c thực dương chứng minh:

$$\sum_{a,b,c} \frac{a+b}{b+c} \cdot \frac{a}{2a+b+c} \geq \frac{3}{4}$$

$$a+b=x; b+c=y; c+a=z \to a=\frac{x+z-y}{2}; b=\frac{x+y-z}{2}; c=\frac{y+z-x}{2}$$

$$\sum_{a,b,c} \frac{x}{y} \cdot \frac{x+z-y}{2(x+z)} \geq \frac{3}{4} \to \sum_{a,b,c} \frac{x}{y} \cdot \frac{x+z-y}{x+z} \geq \frac{3}{2} = \sum_{a,b,c} \frac{x}{y}\left(1-\frac{y}{x+z}\right) \geq \frac{3}{2}$$

$$\sum_{a,b,c} \frac{x}{y} \geq \frac{3}{2} + \frac{y}{x+y} + \frac{x}{z+x} + \frac{z}{y+z}$$

Vế trái Cauchy: $\geq 3 \to$ ta cần chứng minh:

$$x, y, z \text{ thực dương}: \frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+x} \leq \frac{3}{2}$$

là đúng

Không mất tính tổng quát giả sử : $x+y+z=1$

Khi đó $\frac{x}{1-z} + \frac{y}{1-x} + \frac{z}{1-y} \leq \frac{3}{2} \to x(1-x)(1-y) + y(1-y)(1-z) + z(1-z)(1-x) \leq \frac{3}{2}(1-x)(1-y)(1-z)$

$$x+y+z-(x^2+y^2+z^2)-(xy+yz+zx)+x^2y+y^2z+z^2x \geq \frac{3}{2}(1-x-y+xy)(1-z)$$

$$1-(x^2+y^2+z^2)-(xy+yz+zx)+x^2y+y^2z+z^2x \geq \frac{3}{2}(1-x-y+xy-z+xz+yz-xyz)$$

$$1-(x^2+y^2+z^2)-(xy+yz+zx)+x^2y+y^2z+z^2x \geq \frac{3}{2}(xy+yz+zx-xyz)$$

$$2+2(x^2y+y^2z+z^2x)+3xyz \geq 5(xy+yz+zx)+2(x^2+y^2+z^2)$$

$$2+2(x^2y+y^2z+z^2x)+3xyz-(xy+yz+zx) \geq 2(x+y+z)^2 = 2$$

$$\to 2(x^2y+y^2z+z^2x)+3xyz \geq xy+yz+zx$$

$$\to 2x^2y+xyz-xy+2y^2z+xyz-yz+2z^2x+xyz-zx \geq 0$$

$$xy(2x+z-1)+yz(2y+x-1)+zx(2z+y-1) \geq 0$$

$$xy(x-y)+yz(y-z)+zx(z-x) \geq 0$$

$$x^2(y-z)+y^2(z-x)+z^2(x-y) \geq 0 \to x^2(y-z)-y^2(y-z)-y^2(x-y)+z^2(x-y)$$
$$=(x-y)(x+y)(y-z)+(z-y)(y+z)(x-y)=(x-y)(y-z)(x-z) \geq 0$$

Giả sử : $x \geq y \geq z \to x-y \geq 0, y-z \geq 0, x-z \geq 0$ *và tương tự với các hoán vị* $\to$ đpcm