The Method of Vieta Jumping

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The *Vieta Jumping* method, sometimes called *root flipping*, is a method for solving a type of Number Theory problems. This method has been used extensively in mathematical competitions, most recently at the 2007 International Mathematical Olympiad. In this article we analyze this method and present some of its application.

The method of Vieta Jumping can be very useful in problems involving divisibility of positive integers. The idea is to assume the existence of a solution for which the statement in question is wrong. Then to consider the given relation as a quadratic equation in one of the variables. Using Vieta's formula, we can display a second solution to this equation. The next step is to show that the new solution is valid and smaller than the previous one. Then by the argument of infinite descent or by assuming the minimality of the first solution we get a contradiction. To illustrate how this method works let us solve three classical problems.

The first problem can be considered historical; it was submitted to the IMO in 1988 by the West Germany. In [1], Arthur Engel wrote the following note about its difficulty:

Nobody of the six members of the Australian problem committee could solve it. Two of the members were Georges Szekeres and his wife, both famous problem solvers and problem posers. Because it was a number theory problem, it was sent to the four most renowned Australian number theorists. They were asked to work on it for six hours. None of them could solve it in this time. The problem committee submitted it to the jury of the 29^{th} IMO marked with a double asterisk, which meant a superhard problem, possibly too hard to pose. After a long discussion, the jury finally had the courage to choose it as the last problem of the competition. Eleven students gave perfect solutions.

Problem 1. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Prove that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

IMO 1988, Problem 6

Solution. Let $k = \frac{a^2+b^2}{ab+1}$. Fix k and consider all pairs (a, b) of nonnegative integers (a, b) satisfying the equation

$$\frac{a^2+b^2}{ab+1} = k,$$

that is, consider

$$S = \left\{ (a, b) \in \mathbb{N} \times \mathbb{N} \mid \frac{a^2+b^2}{ab+1} = k \right\}.$$

We claim that among all such pairs in S, there exists a pair (a, b) such that $b = 0$ and $k = a^2$. In order to prove this claim, suppose that k is not a perfect square and suppose that $(A, B) \in S$ is the pair which minimizes the sum $a + b$ over all such pairs (if there exist more than one such pair in S, choose an arbitrary one). Without loss of generality, assume that $A \geq B > 0$. Consider the equation

$$\frac{x^2 + B^2}{xB + 1} = k,$$

which is equivalent to

$$x^2 - kB \cdot x + B^2 - k = 0$$

as a quadratic equation in x. We know that $x_1 = A$ is one root of this equation. By Vieta's formula, the other root of this equation is

$$x_2 = kB - A = \frac{B^2 - k}{A}.$$

The first equation implies that x_2 is an integer, the second that $x_2 \neq 0$, otherwise, $k = B^2$ would be a perfect square, contradicting our assumption. Also, x_2 cannot be negative, for otherwise,

$$x_2^2 - kBx_2 + B^2 - k \geq x_2^2 + k + B^2 - k > 0,$$

a contradiction. Hence $x_2 \geq 0$ and thus $(x_2, B) \in S$.

Because $A \geq B$, we have

$$x_2 = \frac{B^2 - k}{A} < A,$$

so $x_2 + B < A + B$, contradicting the minimality of $A + B$. □

Problem 2. Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Prove that

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

Solution. Let $k = \frac{x^2+y^2+1}{xy}$. Fix k and consider all pairs (x, y) of positive integers satisfying the equation

$$\frac{x^2 + y^2 + 1}{xy} = k.$$

Among all such pairs (x, y), let (X, Y) be a pair which minimizes the sum $x + y$. We claim that $X = Y$. To prove this, assume, for the sake of contradiction, that $X > Y$.

Consider now the equation

$$\frac{t^2 + Y^2 + 1}{tY} = k.$$

Then

$$t^2 - kY \cdot t + Y^2 + 1 = 0$$

is a quadratic equation in t. We know that $t_1 = X$ is a root of this equation. The other root can be obtained by Vieta's formula, that is,

$$t_2 = kY - X = \frac{Y^2+1}{X},$$

so, in particular, t_2 is a positive integer. Also, since $X > Y \geq 1$, we have

$$t_2 = \frac{Y^2+1}{X} < X,$$

contradicting the minimality of $X + Y$.

Hence, $X = Y$ and thus X^2 divides $2X^2+1$. Hence X^2 also divides 1, so $X = 1$ and thus $k = \dfrac{X^2+Y^2+1}{XY} = 3$. $\square$

Problem 3. Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a = b$.

IMO 2007, Problem 5

Solution. Because $4ab - 1 \mid (4a^2-1)^2$, we also have

$$4ab - 1 \mid b^2(4a^2-1)^2 - (4ab-1)(4a^3b - 2ab + a^2) = a^2 - 2ab + b^2 = (a-b)^2.$$

Assume that there exist distinct positive integers a and b such that $4ab-1 \mid (a-b)^2$. Let $k = \dfrac{(a-b)^2}{4ab-1} > 0$. Fix k and let

$$S = \left\{(a,b) : (a,b) \in \mathbb{Z}^* \times \mathbb{Z}^* \mid \frac{(a-b)^2}{4ab-1} = k\right\}$$

and let (A, B) be a pair in S which minimizes the sum $a + b$ over all $(a,b) \in S$. Without loss of generality assume that $A > B$. Consider now the quadratic equation

$$\frac{(x-B)^2}{4xB-1} = k, \text{ or } x^2 - (2B + 4kB) \cdot x + B^2 + k = 0,$$

which has roots $x_1 = A$ and x_2. From Vieta's formula,

$$x_2 = 2B + 4kB - A = \frac{B^2+k}{A}.$$

This implies that x_2 is a positive integer, so $(x_2, B) \in S$. By the minimality of $A + B$, we get $x_2 \geq A$, that is

$$\frac{B^2+k}{A} \geq A,$$

and therefore $k \geq A^2 - B^2$. Thus

$$\frac{(A-B)^2}{4AB-1} = k \geq A^2 - B^2$$

and it follows that

$$A - B \geq (A+B)(4AB-1) \geq A + B,$$

a contradiction. $\square$

References

[1] Arthur Engel, *Problem-Solving Strategies*, Springer, 1999

[2] Mathlinks, *IMO 1988, Problem 6*, `http://www.mathlinks.ro/Forum/viewtopic.php?p=352683`

[3] Mathlinks, $xy|x^2 + y^2 + 1$, `http://www.mathlinks.ro/Forum/viewtopic.php?t=40207`

[4] Mathlinks, *IMO 2007, Problem 5*, `http://www.mathlinks.ro/Forum/viewtopic.php?p=894656`