An introduction to motivic cohomology

Khoa Bang Pham
Self-study note

Université de Rennes 1

Abstract

This note is my personal attempt towards a deep understanding of motivic cohomology developed by V. Voevodsky.

Contents

1 Intersection theory

1.1 Chow ring

We assume that X is a variety over a field k.

Definition 1.1. Let $Z_d(X)$ be the free abelian group generated by integral closed subschemes of dimension d of X. Let $Z_*(X)$ be the graded group $\bigoplus Z_d(X)$. An element of $Z_*(X)$ is called an *algebraic cycle* on X, an element of $Z_d(X)$ is called an *d-cycle* on X.

Definition 1.2. Let $Z \subset X$ be a closed subscheme whose irreducible components are denoted by $Z_1, ..., Z_n$. We can associate to Z a cycle

$$[Z] \coloneqq n_1 Z_1 + \cdots n_r Z_r,$$

where $n_i = \text{length}_{\mathcal{O}_{X,\eta_i}} \mathcal{O}_{Z,\eta_i}$ are lengths of module in which η_i's denote generic points of Z_i's.

Recall that a meromorphic function on an integral closed subscheme $Z \subset X$ is simply an element of the function field $k(Z)$.

Definition 1.3 (Rational equivalence). We define an equivalence $\sim$ on $Z_*(X)$ by forcing $i_*[div(f)] = 0$ for all closed immersions of integral subschemes $i : Z \longrightarrow X$ and non-zero meromorphic function f on Z. The quotient group $CH_*(X) = Z_*(X)/\sim$ is the *Chow group.* If $\dim_k(X) = n$, we define

$$Z^d(X) \coloneqq Z_{n-d}(X), \quad CH^d(X) \coloneqq CH_{n-d}(X).$$

Definition 1.4 (Intersecting cycles). Let $X \in \mathbf{Sch}_k$ be a regular (or smooth k-variety) k-scheme, i.e., all local rings are regular local rings. Let V, W be two subvarieties such that their intersection $V \cap W$ is proper, i.e., $\dim(V) + \dim(W) - \dim(X) = \dim(V \cap W)$. Suppose that V, W are respectively represented by two sheaves of ideals I, J. Let $Z \subset V \cap W$ be an irreducible component whose the generic point is η. We define the *intersection product* $V \cdot W$ to be

$$V \cdot W \coloneqq \sum \mu(Z; V, W)[Z],$$

where $\mu(Z; V, W)$ is given by the famous Serre's Tor-formula

$$\mu(Z; V, W) = \sum_{i=0}^{\infty} (-1)^i \text{length}_{\mathcal{O}_{X,\eta}} \text{Tor}_i^{\mathcal{O}_{X,\eta}}(\mathcal{O}_{X,\eta}/I, \mathcal{O}_{X,\eta}/J).$$

Remark. (i) The properness condition in topology is called intersecting tranversally. Two closed submanifolds A, B of a given complex (or C^∞) manifold X intersects tranversally iff at each point $x \in X$, one can choose bases of tangent spaces T_xA, T_xB such that their union forms a basis of T_xX. For instance, see the following image.

(ii) In case that $V \cap W$ is not proper, the *Chow's moving lemma* shows that their exists two cycles V', W' such that V and W are rationally equivalent to V' and W', repsecitvely, such that $V' \cap W'$ is proper. One then define $V \circ W$ to be $V' \cdot W'$. Although rational equivalence helps us to define the intersection product, we lose information

We have several operations on the level of Chow groups associated to a morphism of varieties.

Definition 1.5. Let $f : X \longrightarrow Y$ be a proper morphism. Let Z be an integral closed subscheme of X. As f is proper, $f(Z)$ is an integral closed subscheme of Y. We define the *push forward* f_* by the formula

$$f_*([Z]) = \begin{cases} [k(Z) : k(f(Z))][f(Z)] & \dim Z = \dim f(Z), \\ 0 & \text{otherwise.} \end{cases}$$

Remark. I was quite surprised that even though all books on intersection theory define the push-forward operation, not even a single book I had ever looked at proves the extension $k(Z)/k(f(Z))$ is finite if and only if $\dim Z = \dim f(Z)$, making the operation well-defined. So let me finish this undone part. We reduce the problem to the affine case. Assuming that we are given a ring morphism $A \longrightarrow B$ of two k-algebras which are also integral domains. Since our morphism is dominant, we can assume that this morphism is an inclusion $A \hookrightarrow B$, giving another one $\text{Frac}(A) \hookrightarrow \text{Frac}(B)$. The condition $\dim(A) = \dim(B)$ means that $\text{tr.deg}_k\text{Frac}(A) = \text{tr.deg}_k\text{Frac}(B)$ or $\text{Frac}(B)/\text{Frac}(A)$ is an algebraic extension. However, B is a finitely generated A-algebra. Hence $\text{Frac}(B)$ is also a finitely generated $\text{Frac}(A)$-algebra, now we apply Hilbert nullstellensatz to complete the proof.

Another operation is the flat inverse image.

Proposition 1.6. *Let $f : X \longrightarrow S$ be a flat morphism of relative dimension n, i.e., for all $Z \subset S$ irreducible, $f^{-1}(Z)$ is either empty or equidimensional of dimension $\dim(Z) + n$. Then there exists a unique group morphism $f^* : Z_*(S) \longrightarrow Z_*(X)$ such that*

(i) f^* maps $Z_k(S)$ to $Z_{k+n}(X)$,

(ii) for any closed subscheme $Z \subset S$, we have $f^*([Z]) = [f^{-1}(Z)]$.

Moreover, f^* descends to a well-defined operation $f^* : CH_k(S) \longrightarrow CH_{n+k}(X)$.

Proof. We define f^* so that (ii) is satisfied. To finish the proof, we need to prove that this definition extends to the class of all closed subschemes of S. □

Beside flat inverse, we also have the "usual" pull-back, which requires a little more of effort to obtain.

Definition 1.7. Let $f : X \longrightarrow S$ be a morphism in $\mathbf{Sch}_k$ with X smooth over k. For simplicity, we assume both X and Y are integral. Define $Z^n(X)_f$ to be the subgroup of $Z^n(X)$ containing cycles *in good position* with f (a cycle $Z \subset X$ is called in good position with f

Definition 1.8 (External product). Let $X, Y \in \mathbf{Sch}_k$, we have an external product

$$\begin{aligned} \otimes : Z_n(X) \otimes Z_m(X) &\longrightarrow Z_{n+m}(X \times_k Y) \\ (Z, Z') &\longmapsto [Z \times_k Z'] \end{aligned}$$

and we extend this operation by $\mathbb{Z}$-linearity. This operation descends to an external product

$$\otimes : CH_n(X) \otimes CH_m(Y) \longrightarrow CH_{n+m}(X \times_k Y).$$

Definition 1.9 (Ring structure of Chow groups). Let X be a smooth variety over k. Let $\Delta : X \longrightarrow X \times_k X$ denote the diagonal morphism. The *cup product* is defined by the composition

$$\cup : CH^n(X) \otimes CH^m(X) \xrightarrow{\otimes} CH^{n+m}(X \times_k X) \xrightarrow{\Delta^*} CH^{n+m}(X).$$

We discuss the compatibility of the ring structures with respect to group morphisms induced by morphisms of varieties.

1.2 A crash course on blow-ups and deformation to normal cones

We recall some notion from scheme theory. Let's fix a base scheme X.

Definition 1.10. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There is a *relative spectrum* $f : \mathbf{Spec}_X(A) \longrightarrow X$ such that

$$\mathbf{Spec}_X(A)_{|f^{-1}(U)} = \mathrm{Spec}(\mathcal{A}(U)),$$

where U is an affine open set of X. Being quasi-coherent is a condition imposed to make sure that we can glue all the data $\mathrm{Spec}(\mathcal{A}(U))$ together. Just like the usual spectrum, one has a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X-alg}(\mathcal{A}, \pi_*\mathcal{O}_Y) \simeq \mathrm{Hom}_{\mathrm{Sch}/X}(X, \mathbf{Spec}(\mathcal{A})),$$

where $\pi : Y \longrightarrow X$ is a morphism of scheme.

Definition 1.11. Let $\mathcal{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras which is quasi-coherent and $\mathcal{S}_0 = \mathcal{O}_X$. There is a *global proj* construction, i.e., an X-scheme $p : \mathbf{Proj}(\mathcal{S}) \longrightarrow X$ such that

$$\mathbf{Proj}(\mathcal{S})_{|p^{-1}(U)} = \mathrm{Proj}(\mathcal{S}(U)),$$

where U is an affine open set of X. Being quasi-coherent is a condition imposed to make sure that we can glue all the data $\mathrm{Proj}(\mathcal{S}(U))$ together. If furthermore, S is locally generated by $\mathcal{S}_1$ (after passing to stalks), then $\mathbf{Proj}(\mathcal{S})$ carries an invertible sheaf $\mathcal{O}(1)$.

Example 1.12. Let's fix a scheme X.

(i) It is obvious from the definition that if X is affine, then $\mathbf{Spec}_X(\mathcal{A}) = \mathrm{Spec}(\mathcal{A}(X))$.

(ii) Let $\mathcal{E}$ be a quasi-coherent sheaf on X, then the sheaf of symmetric algebras $\mathbf{Sym}(\mathcal{E})$ is naturally a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules, generated by elements of degree 1. The associated global proj, denoted $\mathbb{P}(\mathcal{E})$, is called the projective bundle. Despite the name, it is not always a vector bundle, it is when $\mathcal{E}$ is a vector bundle.

(iii) Suppose we are given a quasi-coherent graded $\mathcal{O}_X$-algebra $\mathcal{S}$. The *affine cone* and the *projective cone* of R are defined as $C_{\mathcal{S}} := \mathbf{Spec}_X(\mathcal{S})$ and $\mathbb{P}(\mathcal{S}) := \mathbf{Proj}_X(\mathcal{S})$.

(iv) If $Z \hookrightarrow X$ is a closed immersion whose sheaf of ideals is $\mathcal{I}$, the affine cone

$$\mathbf{Spec}_X\left(\bigoplus_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

is called the *normal cone*, often denoted $C_Z X$.

(v) If $\mathcal{S} \longrightarrow \mathcal{R}$ is a surjective morphism of quasi-coherent graded $\mathcal{O}_X$-algebras, then there are closed immersions $\mathbf{Spec}_X(\mathcal{S}) \hookrightarrow \mathbf{Spec}_X(\mathcal{R})$, and $\mathbf{Proj}_X(\mathcal{S}) \hookrightarrow \mathbf{Proj}_X(\mathcal{R})$. In particular, if one assumes that $\mathcal{S}_0 = \mathcal{O}_X$ then the natural projection

$$\mathcal{S} = \mathcal{O}_X \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \twoheadrightarrow \mathcal{O}_X$$

yields the zero section $X \hookrightarrow \mathbf{Spec}_X(\mathcal{S})$.

(vi) Given a quasi-coherent graded $\mathcal{O}_X$-algebra, we can construct a new graded algebra whose n-th degree part is

$$\mathcal{S}_n \oplus \mathcal{S}_{n-1}t \oplus \cdots \oplus \mathcal{S}_0 t^n,$$

where t is an intermediate variable. The affine cone is this new algebra is denoted $C_{\mathcal{S}} \oplus 1$, the projective cone $\mathbb{P}(C_{\mathcal{S}} \oplus 1)$ is called the *projective completion*. For a vector bundle $\mathcal{E}$, one can also define $\mathbb{P}(E \oplus 1)$.

Definition 1.13 (Blow-up). Let $Z \hookrightarrow X$ be a closed immersion of schemes. A *blow-up of X along Z* is a pair $(\mathrm{Bl}_Z X, \pi)$ where $\mathrm{Bl}_Z X$ is a scheme and $\pi : \mathrm{Bl}_Z X \longrightarrow X$ is a morphism of schemes such that $\pi^{-1}(Z)$ is an effective Cartier divisor and this pair is required to be universal with this property. The preimage $\pi^{-1}(Z)$, denoted $E_Z X$, is called the *exceptional divisor*.

Let $Z \hookrightarrow X$ be a closed immersion of schemes, and let $\mathcal{I} \subset \mathcal{O}_X$ be its corresponding sheaf of ideals (which is quasi-coherent). The $\mathcal{O}_X$-algebra $\oplus_{k \geqslant 0} \mathcal{I}^k$ is a graded coherent $\mathcal{O}_X$-algebra that is generated in degree 1.

Proposition 1.14. *The global proj* $\mathbf{Proj}(\oplus_{k\geqslant 0}\mathcal{I}^k)$ *together with the structural morphism* $\mathbf{Proj}(\oplus_{k\geqslant 0}\mathcal{I}^k) \longrightarrow X$ *is a blow-up of* X *along* Z *whose exceptional divisor is the projective cone* $\mathbf{Proj}(\oplus_{k\geqslant 0}\mathcal{I}^k/\mathcal{I}^{k+1})$.

Proof. See [Har77] or [GW10]. □

Definition 1.15. A closed immersion $Z \longrightarrow X$ is called a *regular embedding of codimension* d if locally $X = \mathrm{Spec}(A), Z = \mathrm{Spec}(A/I)$, then I can be generated by a regular sequence of length d.

Example 1.16. (i) Composition of regular imbeddings is a regular embedding.

(ii) Any hypersurface H in $\mathbb{P}^n_k$ (or $\mathbb{A}^n_k$) defines a regular immersion $H \longrightarrow \mathbb{P}^n_k$ of codimension 1.

(iii) If X, Z are smooth k-varieties, then any embedding $Z \longrightarrow X$ is regular.

Remark. The advantage of being a regular embedding is that the *conormal sheaf* $\mathcal{I}/\mathcal{I}^2$ is locally free (thus a vector bundle, whose rank equals the length of defining regular sequences) and the natural map $\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2) \longrightarrow \bigoplus_{n=0}^{\infty}\mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism. In other words, the normal cone $\mathbf{Spec}\left(\bigoplus_0^{\infty}\mathcal{I}^n/\mathcal{I}^{n+1}\right)$ coinsides with the normal bundle $N_Z X \coloneqq (\mathcal{I}/\mathcal{I}^2)^{\vee}$. For more details, one consults [Ful98], **Appendices A6, B7**.

Now we come to the goal of this section, namely, the deformation theory to normal cones. Roughly, let $Z \hookrightarrow X$ be a closed immersion, we can deform X into $C_Z X$ as a zero section. This deformation plays the role of a tubular neighborhood in differential topology.

Proposition 1.17. *Suppose that* $i : Z \longrightarrow X$ *is a closed immersion. There is a flat family* $\pi : M \longrightarrow \mathbb{P}^1_Z$ *with generic fiber* X *and special fiber* $C_Z X$ *such that there exists a family of closed embeddings* $X \times \mathbb{P}^1 \longrightarrow M$ *over* $\mathbb{P}^1_Z$ *such that*

(i) Over any point $t \in \mathbb{P}^1_Z \setminus \{0\}$, *the associated embeddings are an embedding* $Z \times \{t\} \hookrightarrow X$.

(ii) The fiber over $0 \in \mathbb{P}^1$ *is an embedding of* $Z \hookrightarrow C_Z X$ *given by the zero section.*

Proof. For a proof of this proposition, I found Vakil's course on intersection theory is the most readable one, self-contained and very short, try the link. The scheme M can be constructed explicitly as $M = \mathrm{Bl}_{Z\times 0}(X \times \mathbb{P}^1)$. Away from $t = 0$ (t: coordinate on $\mathbb{P}^1$), we do not do anything. Over $t = 0$, we glue $\mathrm{Bl}_Z(X)$ with $\mathbb{P}(C_Z X \oplus 1)$ along the exception divisor $E_Z X$. □

2 The category of Chow motives

Let k be a field. We denote by $\mathbf{SmProj}_k$ the category of projective smooth varieties over k.

Definition 2.1. We define a category $\mathbf{Corr}(k)$ whose objects are objects of $\mathbf{SmProj}_k$ and morphisms are given by

$$\mathrm{Corr}^r(X, Y) \coloneqq \bigoplus_i Z^{\dim(X_i)+r}(X_i \times_k Y),$$

where $X = \coprod X_i$ is the decomposition of X into irreducible components, we call elements of $\mathrm{Corr}^r(X, Y)$ r-*correspondences*. Of course we can compose correspondences. Let $\alpha \in \mathrm{Corr}^r(X, Y)$ and $\beta \in \mathrm{Corr}^s(Y, Z)$, their composition is given by

$$\beta \circ \alpha \coloneqq \pi_{XZ*}(\pi^*_{XY}(\alpha) \cdot \pi^*_{YZ}(\beta)),$$

where $\pi_{XZ}, \pi_{XY}, \pi_{YZ}$ are projections from $X \times_k Y \times_k Z$ onto corresponding factors and $\cdot$ denotes the intersection product.

Remark. (i) Given any morphism $f : X \longrightarrow Y$ in $\mathbf{SmProj}_k$, the associated graph Γ_f is a correspondence and one checks that for composable morphisms f, g, we have $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{g\circ f}]$. This defines a *contravariant* functor

$$\begin{aligned} h : \mathbf{SmProj}_k &\longrightarrow \mathbf{Corr}(k) \\ X &\longmapsto X \\ f &\longmapsto \Gamma_f. \end{aligned}$$

(ii) It is somehow difficult to image why this should gives rise to a well-defined composition, making $\mathbf{Corr}(k)$ a category. Let me illustrate this by a heuristic argument. Correspondences are indeed a generalization of multi-valued functions. It has been noticed for a long time ago that we can think of multi-valued functions as normal functions. This idea is traced back to Riemann when he introduced the notion of Riemann surfaces. For instance, the complex function $z = f(w) = z^{1/n}$ is not really a function unless we choose a branch-cut. We can do other ways around, one of them is to consider the graph $\left\{(z, w) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid z^n = w\right\}$ which projects to $\mathbb{P}^1$ in the w coordinate with generically n preimages. In other words, by identifying a multi-valued function with its graph, we enlarged the class of functions. Similarly, by considering correspondences, we enlarged the class of morphisms among varieties. Now suppose we are given two multi-valued function $f : X \longrightarrow Y, g : Y \longrightarrow Z$, identified with their graphs, how can we compose them? We can simply take the intersection of products $\Gamma_f \times Z \cap X \times \Gamma_g$, and then pushing forward the intersection to $X \times Z$, i.e.,

$$\{(x, f(x)\} \times Z \cap X \times \{(y, g(y))\} = \{(x, f(x), (g \circ f)(x)\} \longmapsto \{x, (g \circ f)(x)\}\,.$$

This is really what we did in the formula $\beta \circ \alpha := \pi_{XZ*}(\pi^*_{XY}(\alpha) \cdot \pi^*_{YZ}(\beta))$. For a concrete proof (ugly one!), one can have a look at [Ful98], **Proposition 16.1.1**.

The following is trivial

Lemma 2.2. *The category* $\mathbf{Corr}(k)$ *has direct sums (on the level of objects) and tensor products given by*

$$X \oplus Y := X \coprod Y, \quad X \otimes Y := X \times_k Y.$$

The sum of morphisms is given by

$$\alpha + \beta := (\alpha, \beta) \in Z^*(X \times_k X) \oplus Z^*(Y \times_k Y) \hookrightarrow Z^* \left((X \coprod Y) \times (X \coprod Y)\right).$$

This makes $\mathbf{Corr}(k)$ *into a preadditive category.*

Definition 2.3 (Pseudoabelian category). A *pseudoabelian category* is a preadditive category such that every idempotent morphism has a kernel. An elementary argument shows that every idempotent has a cokernel.

Given an arbitrary category $\mathcal{C}$, there is a process called *Karoubian envelope* or *pseudoabelianization*.

Lemma 2.4. *Given an arbitrary category, there exists a category* $\mathrm{Kar}(\mathcal{C})$ *together with a functor* $\delta : \mathcal{C} \longrightarrow \mathrm{Kar}(\mathcal{C})$ *such that the image* $\delta(p)$ *of every idempotent* p *in* $\mathcal{C}$ *splits in* $\mathrm{Kar}(\mathcal{C})$*. Moreover, if* $\mathcal{C}$ *is preadditive, the functor* $\delta : \mathcal{C} \longrightarrow \mathrm{Kar}(\mathcal{C})$ *is additive.*

Proof. We can construct $\mathrm{Kar}(\mathcal{C})$ explicitly in the following way: the objects of $\mathrm{Kar}(\mathcal{C})$ are pairs (X, p) where $X \in \mathcal{C}$ and $p : X \longrightarrow X$ is an idempotent. A morphism $f : (X, p) \longrightarrow (Y, q)$ is a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ such that $f = f \circ p = q \circ f$. The natural functor $\mathcal{C} \longrightarrow \mathrm{Kar}(\mathcal{C})$ is given by $X \longmapsto (X, \mathrm{id}_X)$. It is obvious to check that $\mathrm{Kar}(\mathcal{C})$ is the desired category. ☐

Definition 2.5 (Pure effective Chow motives). The *category of pure effective Chow motives* $\mathbf{Chow}^{eff}(k)$ is defined to be the Karoubian envelope of $\mathbf{Corr}(k)$, i.e., $\mathbf{Chow}^{eff}(k) := \mathrm{Kar}(\mathbf{Corr}(k))$.

Remark. (i) In [Mil13], he explained that if one the final category is abelian, one should at least add the images of idempotents. This is right, but I think it is not enough. For me, there is another subtle reason for doing so, or one may say that it is an advantage. In l-adic cohomology, one can calculate the reduced cohomology of the projective line $\mathbb{P}^1_k$, i.e., cohomology of $\mathbf{L} = (\mathbb{P}^1_k, \infty)$ (pointed at infinity). It is an easy task, say, $H^*_l(\mathbb{P}^1_k, \infty) \simeq \mathbb{Z}_l(-1)[-2]$ where $\mathbb{Z}_l(-1)$ is the dual of the l-adic Tate twist $\mathbb{Z}_l(1) = \lim_{n\in\mathbb{N}} \mu_{l^n}(k)$. Informally, the *Lefschetz motive* $\mathbf{L}(1)[+2]$ is invertible with respect to the tensor product in the derived category of l-adic sheaves. Thus, one should at least be able to define $\mathbf{L}$; but there is no such an object in $\mathbf{Corr}(k)$. The advantage of considering the Karoubian envelope is that one adds such an object. Milne did this implicitly in his note. Let's repeat this. A (elementary or prime in Milne's language) cycle of codimension 1 in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is a curve defined by an irreducible polynomial $P(x_0, x_1, y_0, y_1)$ separably homogeneous in each pair (x_0, x_1) and (y_0, y_1). The rational equivalence class of the cycle is determined by the pairs of degrees. Thus,

$$Z^1(\mathbb{P}^1_k \times \mathbb{P}^1_k) \simeq \mathbb{Z} \times \mathbb{Z},$$

with basis the classes of $0 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times 0$.

(ii) Note that $\mathbf{Chow}^{eff}(k)$ is a pseudo-abelian category. The direct sum of effective motives is given by

$$([X],\alpha)\oplus([Y],\beta) \coloneqq ([X\coprod Y],\alpha+\beta).$$

Finally, we should invert the *Lefschetz motive*. J. Ayoub in [Ayo14] noted that every "theory of motives" that does not invert the Tate motive is probably (in fact, surely) a wrong one.

Definition 2.6 (Pure Chow motives). A *Chow motive* is a pair (M,n) where M is a pure effective Chow motive and $n\in\mathbb{Z}$. A morphism $(M,n)\longrightarrow(M',n')$ is an element in the inductive limit

$$\operatorname{colim}_{k\geqslant -\max\{n,n'\}}\operatorname{Hom}_{\mathbf{Chow}^{eff}(k)}(M\otimes\mathbf{L}^{k+n},M\otimes\mathbf{L}^{k+n'})$$

We therefore have the *category of Chow motives* and a fully faithful embedding $\mathbf{Chow}^{eff}(k)\hookrightarrow\mathbf{Chow}(k)$ given by sending M to $(M,0)$.

3 The category of geometric motives of Voevodsky

In order to solve the problem of partially defined composition of correspondences and extend composition law to smooth but non-projective k-varieties, Voevodsky in [MVW06] has introduced the notion of *finite correspondences*. Let us fix a field k as before.

Definition 3.1 (Finite correspondence). Let $X,Y\in\mathbf{Sch}_k$. We define $\mathbf{FiCorr}(X,Y)$ to be the subgroup of $Z(X\times_k Y)$ generated by integral closed subschemes $W\subset X\times_k Y$ such that

(i) The projection $pr_1: W\longrightarrow X$ is finite,

(ii) the image $pr_1(W)\subset X$ is an irreducible component of X.

Elements of $\mathbf{FiCorr}(X,Y)$ are called *finite correspondences* from X to Y.

Fact. Finite correspondences can be composed just like correspondences. As illustrated by an example of the multi-valued function $z^n=w$; finite correspondences seems to be more natural to me than correspondences because at least one should expect the preimage over each point is finite (more exactly, quasi-finite since our example lies in the complex world).

Definition 3.2. Let's define $\mathbf{FiCorr}(k)$ the category whose objects are objects of $\mathbf{Sm}_k$ and morphisms are finite correspondences. Similar to $\mathbf{Corr}(k)$, one has a functor

$$\begin{aligned} h:\mathbf{Sm}_k &\longrightarrow \mathbf{FiCorr}(k)\\ X &\longmapsto X\\ f &\longmapsto \Gamma_f. \end{aligned}$$

Remark. The operation $\times_k$ makes $\mathbf{FiCorr}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathbf{FiCorr}(k))$ is a triangulated category theory.

Definition 3.3. The category $\widehat{\mathbf{DM}}^{eff}_{gm}(k)$ is the localization of $K^b(\mathbf{FiCorr}(k))$, as a triangulated tensor category, by two relations

(i) For $X\in\mathbf{Sm}_k$, we invert $pr_{1*}:[X\times_k\mathbb{A}^1_k]\longrightarrow X$.

(ii) Suppose that $X\in\mathbf{Sm}_k$ is written as $U\cup V$ where U,V are two Zariski open subsets. We invert the canonical morphism
$$\operatorname{Cone}([U\cap V]\longrightarrow[U]\oplus[V])\longrightarrow[X].$$

The category $\mathbf{DM}^{eff}_{gm}$ of *effective geometric motives* is Karoubian envelope of $\widehat{\mathbf{DM}}^{eff}_{gm}(k)$. There is an obvious functor
$$M_{gm}:\mathbf{Sm}_k\longrightarrow\mathbf{DM}^{eff}_{gm}(k).$$

Remark. Obviously, $\mathbf{DM}_{gm}^{eff}(k)$ has the structure of a triangulated tensor category and for any two k-smooth schemes X, Y, there is a canonical isomorphism

$$M_{gm}(X \times_k Y) \simeq M_{gm}(X) \otimes M_{gm}(Y).$$

We call this the *Kunneth formula*.

Definition 3.4. Note that the unit object of the tensor product is $M_{gm}(\mathrm{Spec}(k))$, denoted $\mathbb{Z}$. Any smooth k-scheme X gives rise to a morphism $M_{gm}(X) \longrightarrow \mathbb{Z}$. Since our category is triangulated, there is a distinguished triangle

$$\widetilde{M}_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow \mathbb{Z} \longrightarrow \widetilde{M}_{gm}(X)[1].$$

We call $\widetilde{M}_{gm}(X)$ the *reduced motive* of X. The *Tate object* $\mathbb{Z}(1)$ is defined as $\widetilde{M}_{gm}(\mathbb{P}^1_k)[-2]$. We set $\mathbb{Z}(n) \coloneqq \mathbb{Z}(1)^{\otimes}$. For any object $A \in \mathbf{DM}_{gm}^{eff}(k)$, we define $A(n) \coloneqq A \otimes \mathbb{Z}(n)$.

As pointed out in the remark below definition 2.5, one should invert $\mathbf{L}$ or $\mathbb{Z}(1)$, here we go with $\mathbb{Z}(1)$. Let's speak first in general. Suppose we are given a triangulated tensor category $\mathcal{T}$ and an object $P \in \mathcal{T}$. Generally, one can invert P to obtain a new category, denoted $\mathcal{T}[P^{-1}]$ just like when we invert $\mathbf{L}$ to obtain $\mathbf{Chow}(k)$, but there is a serious problem that the resulting is not always a tensor category though it is a triangulated one. V. Voevodsky in [Voe98] made a brilliant obserse that the category $\mathcal{T}[P^{-1}]$ is a triangulated tensor category if the cyclic permutation of $P \otimes P \otimes P$ equals identity. There is a whole topic on MathOverFlow to discuss this point, see here. We will revisit this issue once we touch the *motivic stable homotopy category* of Morel and Voevodsky. Here we have something stronger, that is the involution $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \longrightarrow \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ is already the identiy. In order to prove this fact, we first show that we can "inject" $\mathbf{Chow}^{eff}(k)$ into $\mathbf{DM}_{gm}^{eff}(k)$.

Proposition 3.5. *There is a natural functor* $\mathbf{Chow}^{eff}(k) \longrightarrow \mathbf{DM}_{gm}^{eff}(k)$ *making the following diagram commutes*

$$\begin{tikzcd}
\mathbf{SmProj}_k \arrow[r] \arrow[d] & \mathbf{Sm}_k \arrow[d, "M_{gm}"] \\
\mathbf{Chow}^{eff}(k) \arrow[r] & \mathbf{DM}_{gm}^{eff}(k)
\end{tikzcd}$$

Proof. It is clearly sufficient to prove that for smooth projective varities X, Y over k, there is a canonical homomorphism

$$CH_{\dim}(X)(X \times Y) \longrightarrow \mathrm{Hom}_{\mathbf{DM}_{gm}^{eff}(k)}(M_{gm}(X), M_{gm}(Y)).$$

Denote by $h_0(X, Y)$ the cokernel of the homomorphism $\mathbf{FiCorr}(X \times \mathbb{A}^1, Y) \longrightarrow \mathbf{FiCorr}(X, Y)$ given by the difference of restrictions to $X \times 0$ and $X \times 1$. One can easily see that

$$\mathbf{FiCorr}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{DM}_{gm}^{eff}(k)}(M_{gm}(X), M_{gm}(Y))$$

factors through $h_0(X, Y)$. On the other hand, by definition of rational equivalent we have a canonical homomorphism

$$h_0(X, Y) \longrightarrow CH_{\dim(X)}(X \times Y)$$

which is an isomorphism by [FV00], **Theorem 7.1**. □

4 Motivic cohomology: Mayer-Vietoris, $\mathbb{A}^1$-homotopy, Gysin sequence

In this section,the motivic cohomology theory is formulated in terms of $\mathbf{DM}_{gm}^{eff}(k)$. V. Voevodsky proved that it has all the properties conjectured by A. Beillinson and S. Lichtembaum in the mid 1980s. Until now, there are at least four formulations of motivic cohomology: as Bloch's higher Chow groups (see [Blo86]), as Zariski (or Nisnevich) hypercohomology of certain complexes of sheaves, a hom-sets in $\mathbf{DM}_{gm}^{eff}$, and a cohomology represented by a motivic Eilenberg-MacLane spectra. They are all equivalent in a reasonable sense, more specially, when one considers only smooth projective varieties over a field. This section chooses the third formulation. Actually, along with the standard conjectures proposed by A. Grothendieck, it is expected to build a category of mixed motives MM such that the motivic cohomology is expressed as certain Ext-group in this category. As before, we fix an underlying field k.

Definition 4.1 (Motivic cohomology). For $X \in \mathbf{Sm}_k$, $q \geqslant 0$, let

$$H^p(X, \mathbb{Z}(q)) \coloneqq \mathrm{Hom}_{\mathbf{DM}_{gm}(k)}(M_{gm}(X), \mathbb{Z}(q)[p]),$$

and call it the (p,q)-*motivic cohomology group*, the index q is called the *weight*. It has a cup product

$$H^p(X, \mathbb{Z}(q)) \otimes H^{p'}(X, \mathbb{Z}(q')) \longrightarrow H^{p+p'}(X, \mathbb{Z}(q+q'))$$

defined as

$$M_{gm}(X) \xrightarrow{\Delta^*} M_{gm}(X) \otimes M_{gm}(X) \xrightarrow{\otimes} \mathbb{Z}(q)[p] \otimes \mathbb{Z}(q')[p'] \simeq \mathbb{Z}(q+q')[p+p']$$

Lemma 4.2 (Mayer-Vietoris sequence). *For $X \in \mathbf{Sm}_k$ and $U, V \subset X$ open subsets, we have a long exact sequence*

$$...H^{p-1}(U \cap V, \mathbb{Z}(q)) \longrightarrow H^p(U \cup V, \mathbb{Z}(q)) \longrightarrow H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \longrightarrow H^p(U \cap V, \mathbb{Z}(q)) \longrightarrow ...$$

Proof. Applying $\mathrm{Hom}_{\mathbf{DM}_{gm}(k)}(\_, \mathbb{Z}(q)[p])$ to the sequence

$$M_{gm}(U \cap V) \longrightarrow M_{gm}(U) \oplus M_{gm}(V) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}(U \cap V)[1]. \qquad \square$$

Lemma 4.3 ($\mathbb{A}^1$-homotopy). *The natural projection $p : X \times_k \mathbb{A}^1_k \longrightarrow X$ induces an isomorphism*

$$p^* : H^p(X \times_k \mathbb{A}^1_k, \mathbb{Z}(q)) \xrightarrow{\sim} H^p(X, \mathbb{Z}(q))$$

for all $X \in \mathbf{Sm}_k$.

Proof. Applying $\mathrm{Hom}_{\mathbf{DM}_{gm}(k)}(\_, \mathbb{Z}(q)[p])$ to the isomorphism $M_{gm}(X \times_k \mathbb{A}^1_k) \simeq M_{gm}(X)$. $\square$

We now come to one of the most important theorem, *the Gysin isomorphism theorem.*

Theorem 4.4 (Gysin isomorphism). *Let $i : Z \hookrightarrow X$ be a closed immersion of codimension n in $\mathbf{Sm}_k$ (in particular, it is a regular imbedding by example 1.16). Then there is a canonical isomorphism in $\mathbf{DM}^{eff}_{gm}(k)$*

$$M_{gm}(X/(X \setminus Z)) \simeq M_{gm}(Z)(n)[2n].$$

Our strategy is that we shall prove the Gysin isomorphism for the case a vector (a morphism $E \longrightarrow X$ locally of the form $pr_1 : U \times \mathbb{A}^n \longrightarrow U$, $U \subset X$ open) first and then reduce the general case by the technique of deforming to normal cones.

Now let $i : Z \hookrightarrow X$ be a closed immersion in $\mathbf{Sm}_k$, as we discuss in the sketch of proof of proposition 1.17, we see that (here we write $\mathbb{P}^1$ with $\mathbb{A}^1 \cup 0$)

$$p : \mathrm{Bl}_{Z \times 0}(X \times \mathbb{A}^1) = X \times (\mathbb{A}^1 \setminus \{0\}) \cup (\mathrm{Bl}_Z X \times_{E_Z X} \mathbb{P}(C_Z X \oplus 1)) \longrightarrow X \times \mathbb{A}^1.$$

We set

$$\mathrm{Def}(i) \coloneqq \mathrm{Bl}_{Z \times 0}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_Z(X).$$

This deformation space deforms i to the zero section of the normal bundle as we showed before.

Proposition 4.5. *The maps*

$$M_{gm}(N_Z X/(N_Z X \setminus s(Z))) \longrightarrow M_{gm}(\mathrm{Def}(i) \setminus Z \times \mathbb{A}^1) \longleftarrow M_{gm}(X/(X \setminus Z))$$

are isomorphisms.

Sketch of proof. This uses a property of Nisnevich topology that a closed imbedding $i : Z \hookrightarrow X$ of codimension is locally isomorphic to the zero section $s_0 : Z \hookrightarrow Z \times \mathbb{A}^d$ (using Nisnevich excision) and then we use the technique of deforming to normal cones. $\square$

Proof of Gysin isomorphism. First, we assume that if $\mathcal{E} \longrightarrow Z$ is a vector bundle of rank n. Then

$$M_{gm}(\mathcal{E}/(\mathcal{E} \setminus s(Z))) \simeq M_{gm}(Z)(n)[2n],$$

where s denotes the zero section. Indeed, $M_{gm}(\mathcal{E}) \simeq M_{gm}(Z)$ is an isomorphism by homotopy invariance, we need to show

$$M_{gm}(\mathcal{E} \setminus s(Z)) \simeq M_{gm}(Z) \oplus M_{gm}(Z)(n)[2n-1].$$

Let $\mathbb{P} \coloneqq \mathbb{P}(\mathcal{E} \oplus 1)$ and write $\mathbb{P} = \mathcal{E} \cup (\mathbb{P} \setminus s(Z))$. Mayer-Vietoris gives a distinguished triangle

$$M_{gm}(\mathcal{E} \setminus s(Z)) \longrightarrow M_{gm}(\mathcal{E}) \oplus M_{gm}(\mathbb{P} \setminus s(Z)) \longrightarrow M_{gm}(\mathbb{P}) \longrightarrow M_{Gm}(\mathcal{E} \setminus s(Z))[1].$$

Since $\mathbb{P} \setminus s(Z) \longrightarrow \mathbb{P}(\mathcal{E})$ is an $\mathbb{A}^1$-bundle, the projective bundle formula gives the desired isomorphism. For the general case, we see that

$$M_{gm}(X/(X \setminus Z)) \simeq M_{gm}(N_Z X/(N_Z X \setminus s(Z))) \simeq M_{gm}(Z)(n)[2n]. \qquad \square$$

We have the following theorem, which is an immediate corollary of the Gysin isomorphism theorem.

Theorem 4.6 (Gysin sequence for motivic cohomology). *Let $i : Z \hookrightarrow X$ be a closed imbedding in $\mathbf{Sm}_k$ with open complement $j : U \hookrightarrow X$. There is a long exact sequence*

$$\ldots \longrightarrow H^{p-1}(U, \mathbb{Z}(q)) \xrightarrow{\partial} H^{p-2n}(Z, \mathbb{Z}(q-n)) \xrightarrow{i_*} H^p(X, \mathbb{Z}(q)) \xrightarrow{j^*} H^p(U, \mathbb{Z}(q)) \longrightarrow \ldots$$

Proof. There is a canonical distinguished sequence

$$M_{gm}(U) \xrightarrow{j^*} M_{gm}(X) \longrightarrow M_{gm}(X/U) \longrightarrow M_{gm}(U)[1],$$

and we insert the Gysin isomorphism to see that

$$M_{gm}(U) \xrightarrow{j^*} M_{gm}(X) \longrightarrow M_{gm}(Z)(n)[2n] \longrightarrow M_{gm}(U)[1].$$

Finally, we apply $\mathrm{Hom}_{\mathbf{DM}_{gm}}(\_, \mathbb{Z}(q)[p])$ to finish the proof. $\square$

Remark. There is a hidden point in the definition here: we first express motivic cohomology as hom set in $\mathbf{DM}_{gm}^{eff}$ but because the natural map $\mathbf{DM}_{gm}^{eff} \longrightarrow \mathbf{DM}_{gm}$ is a fully faithful embedding, it can also be expressed as hom set in $\mathbf{DM}_{gm}$, hence we can cancel out $\mathbb{Z}(q)$'s in the proof above.

5 (Pre)sheaves with transfers

This section study $\mathbf{DM}_{gm}^{eff}(k)$ more deeply, the reader might skip it at the first reading.

Definition 5.1 (Presheaves with transfers). A *presheaf with transfers* is an additive functor $F : \mathbf{Cor}(k)^{op} \longrightarrow \mathbf{Ab}$. The category of presheaves with transfers is denoted $\mathbf{PST}(k)$. A *sheaf with transfer* is a presheaf with transfer such that for each $X \in \mathbf{Sm}_k$, the restriction $F_{|X_{Nis}}$ is a sheaf with respect to the Nisnevich topology. The category of sheaves with transfers is denoted $\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))$. There is an obvious sheafification functor

$$\mathbf{PST}(k) \longrightarrow \mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)), \quad F \longmapsto F_{Nis}$$

left adjoint to the forgetful functor.

Informally, a presheaf with transfers is a presheaf with some extra maps which detect morphisms not coming from graphs of morphisms of schemes.

Example 5.2. (i) The multiplicative presheaf $\mathcal{O}^*$ is a presheaf with transfers.

(ii) Given a smooth k-scheme X, the representable functor $\mathbb{Z}_{tr}(X)(U) = \mathrm{Hom}_{\mathbf{FiCorr}}(U, X)$ is a presheaf with transfers. We denote by $\mathbb{Z}_{tr}(\mathrm{Spec}(k))$ by $\mathbb{Z}$.

(iii) For a pointed k-scheme $(X, x : \mathrm{Spec}(k) \longrightarrow X)$, there is a splitting $\mathbb{Z}_{tr}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x)$. In particular, we are interested in pointed multiplicative group $\mathbb{G}_m \coloneqq (\mathbb{A}^1 - \{0\}, 1)$, which is called the *Tate sphere*. It would show up again when we construct the motivic stable homotopy category.

(iv) Let (X_i, x_i) $(i = \overline{1,n})$, the smash product is defined as

$$\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n) \coloneqq \mathrm{Coker}\left(\mathbb{Z}_{tr}(X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n) \overset{\mathrm{id} \times \cdots \times x_i \times \cdots \times \mathrm{id}}{\longrightarrow} \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)\right).$$

Particularly, we have a presheaf with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$.

Proposition 5.3. *For any smooth k-scheme, $\mathbb{Z}_{tr}(X)$ is a sheaf with transfers in the étale, Zariski and Nisnevich topology.*

Sketch of proof. The proofs are given in [MVW06]. For the Zariski topology, see **Lemma 3.2** in lecture 3. For the étale topology, see **Lemma 6.2** in lecture 6. ☐

Let's discuss the tensor product and the internal hom of the category $\mathbf{PST}(k)$. The objects $\mathbb{Z}_{tr}(X)$ $(X \in \mathbf{Sm}_k)$ are not just sheaves but they are also projective objects in $\mathbf{PST}(k)$. It can be seen easily by Yoneda's lemma. This proves that $\mathbf{PST}(k)$ has enough projectives. Indeed, let $F \in \mathbf{PST}(k)$, then the natural map

$$\bigoplus_{X \in \mathbf{FiCorr}(X), x \in F(X)} \mathbb{Z}_{tr}(X) \longrightarrow F$$

is a surjection. Now if we try do define $(F \otimes G)(X) = F(X) \otimes G(X)$ as a tensor product on $\mathbf{PST}(k)$, then we certainly lose the additivity, i.e., $(F \otimes G)(X \coprod X) = (F(X) \otimes G(X))^{\oplus 4} \neq (F(X) \otimes G(X))^{\otimes 2}$. This suggests that we should find another way around. We proceed as follows. On the class of presheaves of form $\mathbb{Z}_{tr}(X)$, we define

$$\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) \coloneqq \mathbb{Z}_{tr}(X \times Y).$$

In general, let $F, G \in \mathbf{PST}(k)$, and $P_* \longrightarrow F$, $Q_* \longrightarrow G$ be two projective resolutions of F and G whose each degree is a direct sum of some $\mathbb{Z}_{tr}(X)$'s.

Definition 5.4. The *tensor product* and the *internal hom* of F and Q are defined as

$$F \otimes G \coloneqq H_0(\mathrm{Tot}(P_* \otimes Q_*)), \quad \underline{\mathrm{Hom}}(F,G)(X) \coloneqq \mathrm{Hom}(F \otimes \mathbb{Z}_{tr}(X), G),$$

where Tot denotes the total complex of a double complex. With this definition, one can prove tht

$$\mathrm{Hom}(F, \underline{\mathrm{Hom}}(G,H)) \simeq \mathrm{Hom}(F \otimes G, H)$$

for all $F, G, H \in \mathbf{PST}(k)$. The unit of tensor product is undoubtedly $\mathbb{Z}_{tr}(\mathrm{Spec}(k))$.

In order to make computations of morphisms in $\mathbf{DM}_{gm}^{eff}(k)$, Voevodsky has introduced a sheaf-theoretic construction, leading to the category $\mathbf{DM}_{-}^{eff}(k)$. We present here the construction of $\mathbf{DM}_{-}^{eff}(k)$ and the statement of the embedding theorem

Definition 5.5. Let F be a preshef of abelian groups on $\mathbf{Sm}_k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}_k$, the natural projection

$$p^* : F(X) \longrightarrow F(X \times \mathbb{A}^1)$$

is an isomorphism. We call F *strictly homotopy invariant* if for all $q \geqslant 0$, the cohomology presheaf $X \longmapsto H^q(X_{Nis}, F_{Nis})$ is homotopy invariant.

The following is an important theorem of Voevodsky, it is a corollary of the so-called Voevodsky's moving lemma.

Theorem 5.6. *Let F be a homotopy invariant presheaf with transfers on $\mathbf{Sm}_k$. Then*

(i) The cohomology presheaves $X \longmapsto H^q(X_{Nis}, F_{Nis})$ are presheaves with transfers.

(ii) F_{Nis} is strictly homotopy invariant.

(iii) $F_{Zar} = F_{Nis}$ and $H^q(X_{Zar}, F_{Zar}) = H^q(X_{Nis}, F_{Nis})$.

Proof. See [Voe00], **Chapter 3**, **Theorem 4.27 + 5.7**. ☐

Definition 5.7. Inside the derived category $D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$, we have the full-subcategory $\mathbf{DM}_-^{eff}(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant. Moreover, it is *triangulated subcategory* of $D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$.

There is an alternative description of $\mathbf{DM}_-^{eff}(k)$ as a localization of $D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$ rather than a subcategory. For the purpose of stating the localization, we need to introduce the localized map. We define a map from $\mathbf{Comp}^-$ to $\mathbf{DM}_-^{eff}(k)$ first and this map descends to a map on derived category. This map associates each presheaf with transfers an complex analogous to the singular complex of a topological space. There are schemes of the form

$$\Delta^n \coloneqq \operatorname{Spec}\left(\frac{k[x_0,...,x_n]}{x_0+\cdots+x_n-1}\right).$$

This gives a cosimplicial scheme whose faces maps $\partial_j : \Delta^n \longrightarrow \Delta^{n+1}$ given by $x_j \longmapsto 0$. Let F be a presheaf with transfers, we can define new presheaves with transfers

$$F^{\Delta^n}(U) \coloneqq F(U\times\Delta^n).$$

These presheaves fit into a complex

$$C_*F : ... \longrightarrow F^{\Delta^n} \longrightarrow ... \longrightarrow F^{\Delta^2} \longrightarrow F^{\Delta^1} \longrightarrow F \longrightarrow 0.$$

Indeed, it is routine to check that the long sequence above is a complex.

Definition 5.8. The *algebraic singular homology* (one may also call it *homotopy invariant presheaf*) of a presheaf with transfers is defined as the homology of the above complex, i.e., $H^i(F) \coloneqq H_i(C_*F)$. The homotopy invariant presheaf of pointed smooth k-scheme X is the homology of its associated presentable presheaf, i.e., $H_i(C_*\mathbb{Z}_{tr}(X))$. We define $H_i(X/k) = H_i(C_*\mathbb{Z}_{tr}(X))(\operatorname{Spec}(k))$.

Example 5.9. In this example, we compute $H_i(\operatorname{Spec}(k)/k)$. By definition

$$C_*\mathbb{Z}_{tr}(\operatorname{Spec}(k)) : ... \longrightarrow \mathbf{FiCorr}(\Delta^n\times\operatorname{Spec}(k),\operatorname{Spec}(k)) \longrightarrow ... \longrightarrow \mathbf{FiCorr}(\operatorname{Spec}(k),\operatorname{Spec}(k)) \longrightarrow 0.$$

But notice that the only subscheme of $\Delta^n\times\operatorname{Spec}(k)\simeq\Delta^n$ which is finite and surjective over Δ^n is Δ^n, we see that above complex is isomorphic to

$$... \longrightarrow \mathbb{Z}\Delta^n \longrightarrow ... \longrightarrow \mathbb{Z}\Delta^0 \longrightarrow 0.$$

Hence the homology groups are given by

$$H_i(\operatorname{Spec}(k)/k) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{otherwise.}\end{cases}$$

The above computation is essentially as same as the one in topology when computing homology of one-point space.

Remark. (i) If F is a presheaf (resp. sheaf) with transfers then $C_*(F)$ is a complex of presheaves (resp. sheaves) with transfers.

(ii) The homology presheaves $H_i(C_*F)$ are homotopy invariant. Hence, by Voevodsky's theorem, the associated sheaves $H_i^{Nis}(C_iF)$'s are strictly homotopy invariant. We thus have a functor

$$C_* : \mathbf{PST}(k) \longrightarrow \mathbf{DM}_-^{eff}(k).$$

This functor extends to (by taking double complex) to a functor

$$C_* : \mathbf{Comp}^-(\mathbf{PST}(k)) \longrightarrow \mathbf{DM}_-^{eff}(k),$$

which even factors through $\mathbf{Comp}^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$, i.e.,

$$\mathbf{Comp}^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))) \longrightarrow \mathbf{DM}_-^{eff}(k).$$

Theorem 5.10 (Localization theorem). *The functor* $C_* : \mathbf{Comp}^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))) \longrightarrow \mathbf{DM}_-^{eff}(k)$ *descends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))) \longrightarrow \mathbf{DM}_-^{eff}(k),$$

left adjoint to the inclusion $\mathbf{DM}_-^{eff}(k) \subset D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$*. Moreover,* $\mathbf{R}C_*$ *identifies* $\mathbf{DM}_-^{eff}(k)$ *with the localization of* $D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$ *by the full subcategory generated by complexes*

$$\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \longrightarrow \mathbb{Z}_{tr}(X),\ X \in \mathbf{Sm}_k.$$

Proof. See [Voe00], **Proposition 3.2.3**. □

The next significant theorem is the embedding theorem. Start with the functor

$$\mathbb{Z}_{tr} : \mathbf{FiCorr}(k) \longrightarrow \mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)),$$

sending a smooth k-scheme to the representable sheaf $L(X)$. L extends to the homotopy category of bounded complexes

$$\mathbb{Z}_{tr} : K^b(\mathbf{FiCorr}(k)) \longrightarrow D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))).$$

Theorem 5.11. *There is a commutative diagram of exact tensor functors*

$$\begin{tikzcd}
K^b(\mathbf{FiCorr}(k)) \arrow[r, "\mathbb{Z}_{tr}"] \arrow[d] & D^-(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))) \arrow[d, "\mathbf{R}C_*"] \\
\mathbf{DM}_{gm}^{eff}(k) \arrow[r, "i"'] & \mathbf{DM}_-^{eff}(k)
\end{tikzcd}$$

such that i *is a full embedding with dense image and* $\mathbf{R}C_*(\mathbb{Z}_{tr}(X)) \simeq C_*(X)$*.*

As an immediate corollary, one has

Corollary 5.12.

$$\mathrm{Hom}_{\mathbf{DM}_{gm}^{eff}}(M_{gm}(X), M_{gm}(Y)[n]) \simeq \mathbb{H}^n(Y_{Nis}, C_*(X)) \simeq \mathbb{H}^n(Y_{Zar}, C_*(X))$$

for all $X, Y \in \mathbf{Sm}_k$*.*

6 Motives with compact support and duality

Now we reach the part of motives with compact support and duality. Throughout this section, we assume that k is a field admitting a resolution of singularities.

Definition 6.1. Let $X \in \mathbf{Sch}_k$ (category of scheme of finite type over k), $\mathbb{Z}_{tr}^c(X)$ be the presheaf with transfers with $\mathbb{Z}_{tr}^c(X)(U)$ the free abelian group on irreducible $W \subset U \times X$ such that $W \longrightarrow U$ quasi-finite, and dominant onto some component of U.

Thus, we have a pair of functors

$$C_* : \mathbf{Sch}_k \longrightarrow \mathbf{DM}_-^{eff}(k), \quad C_*^c : \mathbf{Sch}_k' \longrightarrow \mathbf{DM}_-^{eff}(k),$$

where $\mathbf{Sch}_k' \subset \mathbf{Sch}_k$ is the subcategory with the same objects as $\mathbf{Sch}_k$ with only *proper* morphisms. Here are three main theorems involving C_*^c.

Theorem 6.2. *Let* $U \cup V = X$ *be an open cover of* $X \in \mathbf{Sch}_k$*. There is a canonical distinguished sequence in* $\mathbf{DM}_-^{eff}(k)$

$$C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*(X) \longrightarrow C_*(U \cap V)[+1].$$

Proposition 6.3. *Let* $p \sqcup i : Y \sqcup F \longrightarrow X$ *be an abstract blow-up. There is a canonical distinguished triangle in* $\mathbf{DM}_-^{eff}(k)$

$$C_*(p^-1(Z)) \longrightarrow C_*(Y) \oplus C_*(F) \longrightarrow C_*(X) \longrightarrow C_*(p^{-1}(Z))[+1].$$

Theorem 6.4. *Let $i : Z \hookrightarrow X$ be a closed immersion whose open complement is $j : U \hookrightarrow X$. There is a distinguished triangle in $\mathbf{DM}_{-}^{eff}(k)$*

$$C_*^c(Z) \longrightarrow C_*^c(X) \longrightarrow C_*^c(U) \longrightarrow C_*^c(Z)[+1].$$

Corollary 6.5. *Both functors C_*, C_*^c factor canonically through the embedding $i : \mathbf{DM}_{gm}^{eff}(k) \longrightarrow \mathbf{DM}_{-}^{eff}(k)$.*

Proofs of four statements. See [Lev13], **Section 9.3**. □

Definition 6.6 (Motives (with compact support)). Let

$$M_{gm} : \mathbf{Sch}_k \longrightarrow \mathbf{DM}_{gm}^{eff}(k), \quad M_{gm}^c : \mathbf{Sch}_k' \longrightarrow \mathbf{DM}_{gm}^{eff}(k)$$

be functors with $i \circ M_{gm} = C_*$ and $i \circ M_{gm}^c = C_*^c$. For $X \in \mathbf{Sch}_k$, we call $M_{gm}(X)$ the *motive* of X, while $M_{gm}^c(X)$ is called the *motive with compact support* of X.

Just like the embedding $\mathbf{Chow}^{eff}(k) \hookrightarrow \mathbf{Chow}(k)$, we have a fully faithful embedding $\mathbf{DM}_{gm}^{eff}(k) \hookrightarrow \mathbf{DM}_{gm}(k)$. This deserves a name, the *cancellation theorem*. For more details, see [Lev13].

Theorem 6.7. *For any objects A, B in $\mathbf{DM}_{gm}^{eff}(k)$, the natural map*

$$- \otimes \mathrm{id} : \mathrm{Hom}_{\mathbf{DM}_{gm}^{eff}(k)}(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{DM}_{gm}^{eff}(k)}(A(1), B(1))$$

is an isomorphism. Thus the natural map

$$i : \mathbf{DM}_{gm}^{eff}(k) \longrightarrow \mathbf{DM}_{gm}(k)$$

is a full embedding.

Now we discuss duality a little bit. The category $\mathbf{DM}_{gm}(k)$ is not solely a tensor triangulated category but even a rigid one, i.e., every object is strong dualizable. The internal hom on $\mathbf{PST}(k)$ provides us an internal hom for $\mathbf{Comp}^{-}(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k)))$ to which we can right-derive to a hom

$$\mathbf{R}\underline{\mathrm{Hom}}(-,-) : D^{-}(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))) \times D^{-}(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))) \longrightarrow D^{-}(\mathbf{Sh}^{Nis}(\mathbf{FiCorr}(k))).$$

This right-derived functor descend to $\mathbf{DM}_{-}^{eff}(k)$. Now if $A, B \in \mathbf{DM}_{gm}^{eff}(k)$, we write

$$\underline{\mathrm{Hom}}_{\mathbf{DM}_{-}^{eff}(k)}(A, B) \coloneqq \mathbf{R}\underline{\mathrm{Hom}}(i(A), i(B)).$$

It is proven that for $n >> 0$, $\underline{\mathrm{Hom}}_{\mathbf{DM}_{-}^{eff}(k)}(A, B(m)) \in \mathbf{DM}_{gm}^{eff}(k)$ and natural maps

$$\underline{\mathrm{Hom}}_{\mathbf{DM}_{-}^{eff}(k)}(A, B(m))(N) \in \mathbf{DM}_{gm}^{eff}(k) \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{DM}_{-}^{eff}(k)}(A(n), B(n+m+N)) \in \mathbf{DM}_{gm}^{eff}(k)$$

are isomorphisms for all n, m and $N = N(A, B)$ (not easy, it makes a heavy use of bivariant cycle cohomology). If $A, B \in \mathbf{DM}_{gm}(k)$, we define

$$\underline{\mathrm{Hom}}_{\mathbf{DM}_{gm}(k)}(A, B) \coloneqq \underline{\mathrm{Hom}}_{\mathbf{DM}_{-}^{eff}(k)}(A(n), B(n+N))(-N).$$

for n, N large enough such that $A(n), B(n) \in \mathbf{DM}_{gm}^{eff}(k)$. Our aforementioned claim asserts that this definition is independent of the choice of n, N. Finally, we set $A^* \coloneqq \underline{\mathrm{Hom}}(A, \mathbb{Z})$.

Theorem 6.8. *(i) For $A \in \mathbf{DM}_{gm}(k)$, the canonical map $A \longmapsto (A^*)^*$ is an isomorphism.*

(ii) For $A, B \in \mathbf{DM}_{gm}(k)$, there are canonical isomorphisms

$$(A \otimes B)^* \simeq A^* \otimes B^*, \quad \underline{\mathrm{Hom}}(A, B) \simeq A^* \otimes B.$$

(iii) For $X \in \mathbf{Sm}_k$ of pure dimension n, one has

$$M_{gm}(X) = M_{gm}^c(X)(-n)[-2n], \quad M_{gm}^c(X) = M_{gm}(X)(-n)[-2n].$$

After all things we have discussed so far, one may thought that the category $\mathbf{DM}_{gm}^{eff}(k)$ is a good candidate for the category of mixed motives conjectured by A. Grothendieck (at least when k admits a resolution of singularities). Sadly, it fails. To the outsider observer, the category of mixed motives MM is an abelian tensor category together with a functor $\mathbf{Sch}_k \longrightarrow MM$ such that motivic cohomology can be expressed as $\mathrm{Ext}_{MM}^*(1, ?)$. P. Deligne suggested that one can construct $D^b(MM)$ first and then go back to MM by the so-called t-structure. V. Voevodsky however, proved that $\mathbf{DM}_{gm}^{eff}(k)$ does not have a reasonable t-structure (that means it is not the derived category of an abelian category). For more details, see [Voe00], **Proposition 4.3.8**.

7 Motivic cohomology: computation in low dimensions, projective bundle formula

In this section, we formulate motivic cohomology in a second way as certain hypercohomology groups and show how it is related to some classical groups. The reason for this is I found out that motivic cohomology has a lot of interesting properties which is somehow intricate to study by using an unique formulation (there are four formulations til now) and therefore for each group of properties, I choose the quickest formulation leading to them. Thus, I do not intend to show how these all formulations agree.

Definition 7.1 (Motivic cohomology). For every integer $q \geqslant 0$, the *motivic complex* $\mathbb{Z}(q)$ is defined as the following complex of presheaves with transfers

$$\mathbb{Z}(q) \coloneqq C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

If A is any abelian group, we define $A(q) = \mathbb{Z}(q) \otimes A$. The *motivic cohomology groups* $H^{p,q}(X, A)$ are defined to be the hypercohomology of motivic complexes $\mathbb{Z}(q)$ with respect to the Zariski topology

$$H^{p,q}(X, A) \coloneqq \mathbb{H}^p_{Zar}(X, A(q)).$$

Remark. It takes us some effort to see the cup product of the definition above. We, however, treat the cup product as a black box in this definition. One may convince himself that we already defined the cup product in the first formulation (definition 4.1).

Theorem 7.2. *There is a quasi-isomorphism of complexes of presheaves with transfers* $\mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$.

Sketch of proof. As opposed to its statement, the proof is highly nontrivial. We consider the functor $\mathscr{M}(\mathbb{P}^1; 0, \infty) : \mathbf{Sm}_k \longrightarrow \mathbf{Ab}$ which sends a smooth k-scheme X to the group of rational functions on $X \times \mathbb{P}^1$ which are regular in a neighborhood of $X \times \{0, \infty\}$ and equal 1 on $X \times \{0, \infty\}$. Then for any $f \in \mathscr{M}^*(\mathbb{P}^1; 0, \infty)(X)$, the associated Weil divisor $D(f)$ belongs to $\mathbf{FiCorr}(X, \mathbb{A}^1 - \{0\})$ and we have a short exact sequence

$$0 \longrightarrow \mathscr{M}^*(\mathbb{P}^1; 0, \infty) \longrightarrow \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\})(X) \longrightarrow \mathbb{Z} \oplus \mathcal{O}^*(X) \longrightarrow 0.$$

Moreover, $\mathscr{M}^*(\mathbb{P}^1; 0, \infty)$ is a presheaf with transfers and $C_*(\mathscr{M}^*)(X)$ is acyclic for any smooth k-scheme X, yielding the result. ☐

Corollary 7.3. *Let* $X \in \mathbf{Sm}_k$, *then*

$$H^{p,q}(X, \mathbb{Z}) = \begin{cases} 0 & q \leqslant 1 \text{ and } (p,q) \neq (0,0), (1,1), (2,1) \\ \mathbb{Z}(X) & (p,q) = (0,0) \\ \mathcal{O}^*(X) & (p,q) = (1,1) \\ \operatorname{Pic}(X) & (p,q) = (2,1). \end{cases}$$

Suppose l is a prime such that $1/l \in k$. Tensoring the isomorphism $\mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$ with $\mathbb{Z}/l(1)$ giving

Corollary 7.4. *If* $1/l \in k$ *and* X *smooth over* k, *then* $H^p(X, \mathbb{Z}/l(1)) = 0$ *for* $p \neq 0, 1, 2$ *while*

$$\begin{aligned} H^{0,1}(X, \mathbb{Z}/l) &= \mu_l(X) \\ H^{1,1}(X, \mathbb{Z}/l) &= H^1_{\acute{e}t}(X, \mu_l) \\ H^{2,1}(X, \mathbb{Z}/l) &= \operatorname{Pic}(X)/l\operatorname{Pic}(X). \end{aligned}$$

Definition 7.5. Let $X \in \mathbf{Sm}_k$ and L be a line bundle on X, we define the *first Chern class* $c_1(L)$ of L to be the corresponding element of $[L] \in \operatorname{Pic}(X)$ in $H^{2,1}(X, \mathbb{Z})$.

Now we have enough tools to prove the *projective bundle formula.* Let $E \longrightarrow X$ be a vector bundle of rank $n+1$ with $q \in \mathbf{Sm}_k$. Let $\mathbb{P}(E) \longrightarrow X$ be the resulting $\mathbb{P}^n$ bundle whose the tautological bundle is denoted $\mathcal{O}(1)$. We define a morphism $\alpha_j : M_{gm}(\mathbb{P}(E)) \longrightarrow M_{gm}(X)(j)[2j]$ by

$$M_{gm}(\mathbb{P}(E)) \overset{\text{diagonal}}{\longrightarrow} M_{gm}(\mathbb{P}(E)) \otimes M_{gm}(\mathbb{P}(E)) \overset{q \otimes c_1(\mathcal{O}(1))^j}{\longrightarrow} M_{gm}(X)(j)[2j].$$

The projective bundle is the following

Theorem 7.6 (Projective bundle formula). *The map*

$$\alpha_E = \bigoplus_{j=0}^{n} \alpha_j : M_{gm}(\mathbb{P}(E)) \longrightarrow \bigoplus_{j=0}^{n} M_{gm}(X)(j)[2j]$$

is an isomorphism.

Proof. We divide the proof into several steps

(i) First, $M_{gm}(\mathbb{A}^n \setminus 0) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$. Indeed, if $n = 1$, then $M_{gm}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ by the very definitino of $\mathbb{Z}(1)$. Using the Mayer-Vietoris sequence

$$M_{gm}(\mathbb{A}^1 \setminus 0) \longrightarrow M_{gm}(\mathbb{A}^1) \oplus M_{gm}(\mathbb{A}^1) \longrightarrow M_{gm}(\mathbb{P}^1) \longrightarrow M_{gm}(\mathbb{A}^1 \setminus 0)[1]$$

(because $\mathbb{P}^1 = \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ is an elementary square) to define an isomorphism $M_{gm}(\mathbb{A}^1 \setminus 0) \simeq \mathbb{Z}(1) \oplus \mathbb{Z}$.

(ii) For general n, I claim that $M_{gm}(\mathbb{A}^n \setminus 0) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$. Indeed, we write $\mathbb{A}^n = (\mathbb{A}^n \setminus \mathbb{A}^{n-1}) \cup (\mathbb{A}^n \setminus \mathbb{A}^1)$ and use induction together with Mayer-Vietoris and homotopy invariance, this gives the distinguished triangle

$$(\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}) \longrightarrow (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \oplus (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}) \longrightarrow M_{gm}(\mathbb{A}^n \setminus 0) \longrightarrow (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3]$$

yielding the result.

(iii) Regarding the projective bundle formula, the map α_E is natural in X, E. Mayer-Vietoris reduces to the case of a trivial bundle, then to the case $X = \mathrm{Spec}(k)$, so we have to prove

$$M_{gm}(\mathbb{P}^n) \simeq \bigoplus_{j=0}^{n} \mathbb{Z}(j)[2j].$$

Write $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus 0)$. Since $M_{gm}(\mathbb{A}^n) = \mathbb{Z}$ and $\mathbb{P}^n \setminus 0$ is an $\mathbb{A}^1$-bundle over $\mathbb{P}^{n-1}$, so induction gives

$$M_{gm}(\mathbb{P}^n \setminus 0) = \bigoplus_{j=0}^{n-1} \mathbb{Z}(j)[2j].$$

By our computation, we have $M_{gm}(\mathbb{A}^n \setminus 0) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$. The Mayer-Vietoris distinguished triangle

$$M_{gm}(\mathbb{A}^n \setminus 0) \longrightarrow M_{gm}(\mathbb{A}^n) \oplus M_{gm}(\mathbb{P}^n \setminus 0) \longrightarrow M_{gm}(\mathbb{P}^n) \longrightarrow M_{gm}(\mathbb{A}^n \setminus 0)[1]$$

gives the result. □

8 Motivic cohomology: Relation to Milnor's K-theory and statement of Bloch-Kato conjecture

Let's fix a field F. Let $F^\times = F \setminus 0$.

Definition 8.1 (Milnor's K-theory mod 2). The *Milnor's K ring* $K_*^M(F)$ is the quotient of the tensor algebra $T(F^\times)$ by the ideal generated by elements $a \otimes (1-a)$ with $a \in F \setminus \{0, 1\}$, i.e.,

$$K_*^M(F) := T(F^\times) / \{a \otimes (1-a) \mid a \in F \setminus \{0, \}\}.$$

In this definition, the tensor algebra is taken over $\mathbb{Z}$, and one can see that $K_*^M(F)$ is graded whose the n-th part is

$$K_n^M(F) = T_n(F) / \{a \otimes (1-a) \mid a \in F \setminus \{0, \}\}.$$

Obviously, $K_n^M(F)$ is generated by the classes of elements of the form $a_1 \otimes \cdots \otimes a_n$, we denote by $\{a_1, ..., a_n\}$ these elements. The following list of properties can be proven easily.

Lemma 8.2. *(i)* $\{a_1, ..., a_n\} = 0$ *if* $a_i = 1$ *for some* i.

(ii) $\{a_1, ..., a_i b_i, ..., a_n\} = \{a_1, ..., a_i, ..., a_n\} + \{a_1, ..., b_i, ..., a_n\}$.

(iii) $\{a_1, ..., a_i, 1 - a_i, ..., a_n\} = 0$.

(iv) $\{a, -a\} = 0$.

(v) $\{a, b\} = -\{b, a\}$, *and hence* $\beta\alpha = (-1)^{mn}\alpha\beta$ *for all* $\alpha \in K_n^M(F)$ *and* $\beta \in K_m^M(F)$.

(vi) $\{a_1, ..., a_n\} = 0$ *if* $a_i + a_j$ *equals* 0 *or* 1 *for some pair of distinct indices* i, j.

(vii) *If* $a_1 + \cdots + a_n$ *equals* 0 *or* 1*, then* $\{a_1, ..., a_n\} = 0$.

(viii) $\{a\}^2 = \{a, -1\} = \{-1, a\}$.

Assume that F has a discrete valuation $v : F^\times \longrightarrow \mathbb{Z}$. Since v is a homomorphism, it can be thought of as a morphism $K_1^M F \longrightarrow K_0^M \overline{F}$, where $\overline{F}$ is the residue field. Let's denote by $\mathcal{O}$ the corresponding discrete valuation ring.

Proposition 8.3. *For all* $n \geqslant 1$*, there is a unique morphism, called the residue map,*

$$\partial_v : K_n^M F \longrightarrow K_{n-1}^M \overline{F},$$

satisfying

$$\partial_v(\{a, u_2, ..., u_n\}) = v(a)\{\overline{u_2}, ..., \overline{u_n}\},$$

for all $a \in F^\times$ *and units* $u_2, ..., u_n \in \mathcal{O}^\times$.

Proof. See [Mil70], **Lemma 2.1**. □

If $F \subset E$ is a finite extension, there exists a norm map $N_{E/F} : K_n^M(E) \longrightarrow K_n^M(F)$ which is the multiplication by $[E : F]$ in degree 0 and is the usual norm map in degree 1. The higher norm map is highly trivial to be defined and to be proved that it is canonically defined, i.e., independent of the choice of generators of the field extension. In this document, we can treat it as a black box, it satisfies the *projection law* and *composition law*

$$N_{E/F}(\{\alpha_E, \beta\}) = \{\alpha, N_{E/F}(\beta)\} \quad \text{and} \quad N_{L/F} = N_{L/E} \circ N_{E/F},$$

where $\alpha \in K_n^M(F), \beta \in K_*^M(E)$, and $L/E/F$ is a tower of finite extensions. For more details, one consults [GS17], **Chapter 7**. Later we will see that there is a norm map of motivic cohomology $H^p(E, \mathbb{Z}(q)) \longrightarrow H^p(F, \mathbb{Z}(q))$ given by the proper push-forward having exactly the same properties of the norm map of Milnor's K-theory. We will use this fact to prove that $H^p(E, \mathbb{Z}(p)) = K_p^M(E)$ for any field E.

Theorem 8.4 (Weil reciprocity). *Suppose now that* L *is an algebraic function field over* F*. For each discrete valuation* w *on* L*, let's denote by* $F(w)$ *the residue field* $\overline{L}$*. Then for all* $x \in K_{n+1}^M(L)$*, we have*

$$\sum_w N_{F(w)/F}\partial_w(x) = 0.$$

Corollary 8.5. *Let* $p : Z \longrightarrow \mathbb{A}_F^1$ *be a finite surjective morphism where* Z *is integral. Let* $f_1, ..., f_n \in \mathcal{O}^*(Z)$ *and*

$$p^{-1}(0) = \sum n_i^0 z_i^0 \quad \textit{and} \quad p^{-1}(1) = \sum n_i^1 z_i^0,$$

where n_i^j *is the multiplicity of* $z_i^j = \operatorname{Spec}(E_i^j)$*. Define*

$$\phi_0 = \sum n_i^0 N_{E_i^0/F}(\{f_1, ..., f_n\}_{E_i^0}) \quad \textit{and} \quad \phi_1 = \sum n_i^1 N_{E_i^1/F}(\{f_1, ..., f_n\}_{E_i^1}).$$

Then $\phi_0 = \phi_1$ *in* $K_n^M(F)$.

Proof. Let L be the function field of Z and consider $x = \left\{\frac{t}{t-1}, f_1, ..., f_n\right\}$. At every infinite place, $\frac{t}{t-1}$ equals 1 and $\partial_W(x) = 0$. Similarly, $\partial_w(x) = 0$ at all finite places except those over 0 and 1. If w_i lies over $t = 0$ then $\partial_{w_i}(x) = n_i^0\{f_1, ..., f_n\}$ in $K_n^M(E_i^0)$; if w_i lies over $t = 1$ then $\partial_{w_i}(x) = -n_i^0\{f_1, ..., f_n\}$ in $K_n^M(E_i^1)$. By Weil's reciprocity, $\sum N\partial_{w_i}(x) = \phi_0 - \phi_1$ vanishes in $K_n^M(F)$. □

Now we suppose that $1/2 \in F$. We consider the Galois cohomology $H^*(F, \mathbb{Z}/2\mathbb{Z})$ with the coefficient in $\mathbb{Z}/2\mathbb{Z}$ equipped with the trivial action of $\mathrm{Gal}(F^{sep}/F)$, where F^{sep} is a separable closure of F. There is a short exact sequence of Galois modules

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow (F^{sep})^\times \xrightarrow{(-)^2} (F^{sep})^\times \longrightarrow 1.$$

It induces a sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow F^\times \xrightarrow{(-)^2} F^\times \longrightarrow H^1(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(F, (F^{sep})^\times).$$

The last term vanishes due to the famous Hilbert's 90-th theorem. Hence we have a canonical isomorphism

$$\partial^1 : k_1^M(F) = (F^\times)/(F^\times)^2 \xrightarrow{\sim} H^1(F, \mathbb{Z}/2\mathbb{Z}).$$

In particular, we see that there is a morphism

$$(\partial^1)^{\otimes n} : k_1^M(F)^{\otimes n} \longrightarrow H^1(F, \mathbb{Z}/2\mathbb{Z})^{\otimes n} \longrightarrow H^n(F, (\mathbb{Z}/2\mathbb{Z})^{\otimes n}) = H^n(F, \mathbb{Z}/2\mathbb{Z}).$$

where the second is the cup product of Galois cohomology and the last equality is simply $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$.

Theorem 8.6 (Bass-Tate). *$(\partial^1)^\otimes$ defines a unique morphism of graded rings $\partial : k_*^M(F) \longrightarrow H^*(F, \mathbb{Z}/2\mathbb{Z})$ which is an isomorphism if F is a finite, local, global or real closed field.*

Proof. See [Mil70], **Lemma 6.1** and **Lemma 6.2**. □

The brilliant readers may realize that it is of no harm to replace 2 by any prime invertible in F and we can use étale cohomology instead of Galois cohomology. Let l be a prime such that $1/l \in F$. Consider the Kummer exact sequence

$$1 \longrightarrow \mu_l \longrightarrow \mathbb{G}_m \xrightarrow{(-)^l} \mathbb{G}_m \longrightarrow 1.$$

Taking étale cohomology yields

$$0 \longrightarrow H^0_{\text{ét}}(F, \mu_l) \longrightarrow H^0_{\text{ét}}(F, \mathbb{G}_m) \xrightarrow{.l} H^0_{\text{ét}}(F, \mathbb{G}_m) \xrightarrow{\partial} H^1_{\text{ét}}(F, \mu_l) \longrightarrow H^1_{\text{ét}}(F, \mathbb{G}_m)...$$

By Hilbert's theorem 90, we have $H^1_{\text{ét}}(F, \mathbb{G}_m) = H^1_{Zar}(F, \mathbb{G}_m) = \mathrm{Pic}(F)$ because 1-dimensional vector bundle over F is (isomorphic) just k. Consequently, we have

$$H^0_{\text{ét}}(F, \mathbb{G}_m)/l \simeq H^1_{\text{ét}}(F, \mu_l).$$

Thus, $F^\times/l \simeq H^1_{\text{ét}}(F, \mu_l)$ because $H^0_{\text{ét}}(\mathbb{G}_m) = \mathbb{G}_m(F) = F^\times$. We call either this isomorphism or its lifitng $\partial : F^\times \longrightarrow H^1_{\text{ét}}(F, \mu_l)$ the *Galois symbol* and analogous to the case of Galois cohomology, we take the cup product

$$\partial^n : (F^\times)^{\otimes n} \xrightarrow{\partial^{\otimes n}} H^1_{\text{ét}}(F, \mu_l)^{\otimes n} \xrightarrow{\cup^{\otimes n}} H^n_{\text{ét}}(F, \mu_l^{\otimes n}).$$

One can check easily that ∂^n factors through Milnor's K-theory. But because $1/l \in F$, $\mu_l \simeq \mathbb{Z}/l$ and étale cohomology is l-torsion, so indeed it gives us a map, call the *norm residue map*

$$\partial^n : K_n^M(F)/l \longrightarrow H^n_{\text{ét}}(F, \mu_l^{\otimes n}).$$

This Bloch-Kato conjecture (proved by Voevodsky, which later helps him to win the Fields medal) asserts that this is in fact an isomorphism. The case $l = 2$ is known as the Milnor's conjecture.

Conjecture 8.7 (Bloch-Kato). *Let F be a field. If l is a prime invertible in F, then for all $n \geqslant 0$, the norm residue map $\partial^n : K_n^M(F)/ \longrightarrow H^n_{ét}(F, \mu_l^{\otimes n})$ is an isomorphism.*

In fact, Voevodsky proved that the Bloch-Kato conjecture is equivalent to the following conjecture, which we already prove in some few low-dimensional cases in corollary 7.4.

Conjecture 8.8 (Beillinson-Lichtenbaum). *Let k be a field. If l is a prime invertible in k, then for all smooth k-varieties X, $p \leqslant q$, there is a naturally well-defined isomorphism*

$$H^{p,q}(X, \mathbb{Z}/l) \simeq H^p_{ét}(X, \mu_l^{\otimes q}).$$

Sadly, we cannot go deeper to proofs of these conjectures, they are all incredibly difficult and beyond the scope of any introductory note. We now turn to a much more easier statement connecting the middle motivic cohomology group with Milnor's K-theory.

Theorem 8.9. *Let F be a field, then there is a naturally well-defined isomorphism $H^{n,n}(\mathrm{Spec}(F),\mathbb{Z}) \simeq K_n^M(F)$ for all $n \geqslant 0$.*

In the following, we will construct

Lemma 8.10. *Let F be a field, we have $H^{p,q}(\mathrm{Spec}(F),\mathbb{Z}) = H_{q-p}(C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}(\mathrm{Spec}(F))$ for all p,q.*

Proof. Write A_* for $C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}(\mathrm{Spec}(F))$ so the right side is $H_{q-p}A_* = H^{p-q}A_*$. By definition, the restriction of $\mathbb{Z}(q)$ to $\mathrm{Spec}(F)$ is the chain complex $A_*[-q]$. Since Zariski cohomology on $\mathrm{Spec}(F)$ is just ordinary cohomology, we have

$$H^{p,q}(\mathrm{Spec}(F),\mathbb{Z}) = H^p(A_*[-q]) = H^{p-q}(A_*) = H_{q-p}(A_*).$$ □

Lemma 8.11. *If $F \subset E$ is a finite field extension, then the proper push-forward of cycles induces a map $N_{E/F} : H^{*,*}(\mathrm{Spec}(E),\mathbb{Z}) \longrightarrow H^{*,*}(\mathrm{Spec}(F),\mathbb{Z})$. Moreover, if $x \in H^{*,*}(\mathrm{Spec}(E),\mathbb{Z})$ and $y \in H^{*,*}(\mathrm{Spec}(F),\mathbb{Z})$, then*

(i) $N_{E/F} : H^{0,0}(\mathrm{Spec}(E),\mathbb{Z}) = \mathbb{Z} \longrightarrow \mathbb{Z} = H^{0,0}(\mathrm{Spec}(F),\mathbb{Z})$ is the multiplication by $[E:F]$.

(ii) $N_{E/F} : H^{1,1}(\mathrm{Spec}(E),\mathbb{Z}) = E^* \longrightarrow F^* = H^{1,1}(\mathrm{Spec}(F),\mathbb{Z})$ is the usual norm map.

(iii) (Projection formulas) $N_{E/F}(y_E \cdot x) = y \cdot N_{E/F}(x)$ and $N_{E/F}(x \cdot y_E) = N_{E/F}(x)\cdot y_E$.

(iv) If $F \subset E \subset K$ and K/F is normal, then

$$N_{E/F}(x)_K = [E:F]_{insep} \sum_{j:E\hookrightarrow K} j^*(x) \in H^{*,*}(\mathrm{Spec}(K),\mathbb{Z}).$$

(v) If $F \subset E' \subset E$ then $N_{E/F}(x) = N_{E'/F}(N_{E/E'}(x))$.

By lemma 8.10, we have to define a map from $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}(\mathrm{Spec}(F))$ to $K_n^M(F)$ which composes with the difference of face operators is zero

$$\begin{tikzcd}
\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \arrow[r, "\partial_0-\partial_1"] & \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F)) \arrow[r] \arrow[dr, dashed, "f"'] & H^{n,n}(\mathrm{Spec}(F),\mathbb{Z}) \arrow[d] \\
 & & K_n^M(F)
\end{tikzcd}$$

(i) The construction goes as follows. We see that $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F))$ is a quotient of of the free abelian group generated by the closed points of $(\mathbb{A}_F^1 \setminus 0)^n$, modulo subgroup generated by all points of from $(x_1,...,1,...,x_n)$. If x is a E-point of $(\mathbb{A}_F \setminus 0)^n$, then x defines non-zero elements $\{x_1,...,x_n\}$ of E, yielding an element $\{x_1,...,x_n\}$ of $K_n^M(E)$. Since E/F is finite, we use the norm map $N_{E/F}$ to push this element to $K_n^M(F)$. This gives a map $f : \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}(\mathrm{Spec}(F)) \longrightarrow K_n^M(F)$ inducing a map $\theta : H^{n,n}(\mathrm{Spec}(F),\mathbb{Z}) \longrightarrow K_n^M(F)$.

(ii) Now we construct an inverse map. If x is a F-point of $(\mathbb{A}_F^1 \setminus 0)^n$, then its coordinate defines non-zero elements $x_1,...,x_n$ of F. We write $[x_1 : \cdots : x_n]$ the class of x in $H^{n,n}(\mathrm{Spec}(F),\mathbb{Z})$. We define a map

$$\lambda : T(F^\times) \longrightarrow \bigoplus_n H^{n,n}(\mathrm{Spec}(F),\mathbb{Z})$$
$$a_1 \otimes \cdots \otimes a_n \longmapsto [a_1 : \cdots : a_n].$$

We want to prove that this map factors through $K_n^M(F)$ so it is enough to prove $[a : 1-a] = 0$ because $[a_1 : \cdots : a_n] = [a_1]\cdots[a_n]$. Voevodsky's trick is to prove that there is some integer n such that $n[a:1-a] = 0$ for all $a \in E$ where E is a finite extension of F, then $[a : 1-a] = 0$ for all $a \in F$ and make a reduction to $n = 1$. Actually, we can choose $n = 12$. Let's see why this happens. We consider a finite correspondence from $\mathbb{A}^1$ (parametrized by t) to $X = \mathbb{A}^1 \setminus 0$ (parametrized by x) defied by

$$x^3 - t(a^3+1)x^2 + t(a^3+1)x - a^3 = 0.$$

Let $\omega^2+\omega+1=0$ and $E=F(\omega)$. The fiber of this correspondence over $t=0$ is $a, \omega a, \omega^2 a$ whereas the fiber over $t=1$ consists of a^3 and two sixth roots of 1. Using the embedding $x \longmapsto (x, 1-x)$ of $\mathbb{A}^1 \setminus \{0,1\}$ into X^2, Z yields a correspondence Z' from $\mathbb{A}^1$ to X^2. In $H^{2,2}(\mathrm{Spec}(E), \mathbb{Z})$ we have

$$\partial_0(Z') = [a:1-a]:[\omega a:1-\omega a]+[\omega^2 a:1-\omega^2 a] = [a:1-a^3]+[\omega:(1-\omega a)(1-\omega^2 a)^2]$$

is equal to (by corollary 8.5)

$$\partial_1(Z') = [a^3:1-a^3]+[-\omega:1+\omega]+[-\omega^2:1+\omega^2].$$

Multiplying by 3 eliminiates terms $[\omega:b]$ and noticing that $[-1:1+\omega]+[-1:1+\omega^2] = [-1:(1+\omega)(1+\omega^2)] = [-1:1] = 0$. Thus, $2[a^3:1-a^3]=0$ over E. Applying the norm yields $4[a^3:1-a^3]=0$ over F. Passing to the extension $F(a^{1/3})$ and norming yields $0 = 12[a:1-a]$ over F.

(iii) The construction already implies that $\theta \circ \lambda$ is identity, the proof λ is surjective is somewhat difficult that we cannot line out here.

References

[Ayo14] Joseph Ayoub. A guide to (étale) motivic sheaves. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1101–1124. Kyung Moon Sa, Seoul, 2014. ↑7

[Blo86] Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986. ↑8

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998. ↑5 , ↑6

[FV00] Eric M. Friedlander and Vladimir Voevodsky. Bivariant cycle cohomology. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 138–187. Princeton Univ. Press, Princeton, NJ, 2000. ↑8

[GS17] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology*, volume 165 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. Second edition of [MR2266528]. ↑17

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises. ↑5

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. ↑5

[Lev13] Marc Levine. Six lectures on motives. In *Autour des motifs—École d'été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres/Asian-French Summer School on Algebraic Geometry and Number Theory. Vol. II*, volume 41 of *Panor. Synthèses*, pages 1–141. Soc. Math. France, Paris, 2013. ↑14

[Mil13] James S. Milne. Motives—Grothendieck's dream. In *Open problems and surveys of contemporary mathematics*, volume 6 of *Surv. Mod. Math.*, pages 325–342. Int. Press, Somerville, MA, 2013. ↑6

[Mil70] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/70. ↑17 , ↑18

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. ↑7 , ↑11

[Voe98] Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 579–604, 1998. ↑8

[Voe00] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000. ↑11 , ↑13 , ↑14