Singapore Open Mathematical Olympiad 2009

Senior Section

Tuesday, 2 June 2009 — 0930-1200 hrs

Important:
Answer ALL 35 questions.
Enter your answer sheet provided.
For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer
For the other short questions, write your answer in the answer sheet
No steps are needed to justify your answer
Each question carries 1 mark
No calculators are allowed.

Multiple Choice Questions

Question 1. Suppose that π is a plane and A and B are two points on the plane π. If the distance between A and B is 33 cm, how many lines are there in the plane such that the distance between each line and A is 7 cm and the distance between each line and B is 26 cm, respectively.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinitely many

Question 2. Let $y = (17 - x)(19 - x)(19 + x)(17 + x)$, where x is a real number. Find the smallest posible value of y.

(A) -1296 **(B)** -1295 **(C)** -1294 **(D)** -1293 **(E)** -1292

Question 3. If two real numbers a and b are randomly chosen from the interval $(0, 1)$, find the probability that the equation $x^2 - \sqrt{a}x + b = 0$ has real roots.

(A) $\frac{1}{8}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{16}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

Question 4. If x and y are real numbers for which $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y$.

(A) -3 **(B)** -1 **(C)** 1 **(D)** 3 **(E)** 5

Question 5. In a triangle ABC, $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$. Find the value of $\cos C$.

(A) $\frac{56}{65}$ or $\frac{16}{65}$ **(B)** $\frac{56}{65}$ **(C)** $\frac{16}{65}$ **(D)** $-\frac{56}{65}$ **(E)** $\frac{56}{65}$ or $-\frac{16}{65}$

Question 6. The area of a triangle ABC is 40 cm^2. Points D, E and F are on sides AB, BC and CA, respectively, as shown in the figure below. If $AD = 3$ cm, $DB = 5$ cm, and the area of triangle ABE is equal to the area of quadrilateral $DBEF$, find the area of triangle AEC in cm^2.

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Question 7. Find the value of $\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} + \cdots + \frac{22}{20!+21!+22!}$.

(A) $1 - \frac{1}{24!}$ **(B)** $\frac{1}{2} - \frac{1}{23!}$ **(C)** $\frac{1}{2} - \frac{1}{22!}$ **(D)** $1 - \frac{1}{22!}$ **(E)** $\frac{1}{2} - \frac{1}{24!}$

Question 8. There are eight envelopes numbered 1 to 8. Find the number of ways in which 4 identical red buttons and 4 identical blue buttons can be put in the envelopes such that each envelope contains exactly one button, and the sum of the nimbers on the envelopes containing the blue buttons.

(A) 35 **(B)** 34 **(C)** 32 **(D)** 31 **(E)** 62

Question 9. Determine the number of acute-angled triangles (i.e., all angles are less than 90') in which all angles (in degrees) are positive integers and the largest angle is three times the smallest angle.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Question 10. Let $ABCD$ be a quadrilateral inscribed in a circle with diameter AC, and let E be the foot of perpendicular from D onto AB, as shown in the figure below. If $AD = DC$ and the area of quadrilateral $ABCD$ is 24 cm^2, find the length of DE in cm.

(A) $3\sqrt{2}$ **(B)** $2\sqrt{6}$ **(C)** $2\sqrt{7}$ **(D)** $4\sqrt{2}$ **(E)** 6

Short Questions

Question 11. Find the number of positive divisors of $(2008^3 + (3 \times 2008 \times 2009) + 1))^2$.

Question 12. Suppose that a, b and c are real numbers greater than 1. Find the value of

$$\frac{1}{1 + \log_{a^2b}\left(\frac{c}{a}\right)} + \frac{1}{1 + \log_{b^2c}\left(\frac{a}{b}\right)} + \frac{1}{1 + \log_{c^2a}\left(\frac{b}{c}\right)}.$$

Question 13. Find the remainder when $(1! \times 1) + (2! \times 2) + (3! \times 3) + \cdots + (286! \times 286)$ is divided by 2009.

Question 14. Find the value of

$$(25 + 10\sqrt{5})^{\frac{1}{3}} + (25 - 10\sqrt{5})^{\frac{1}{3}}.$$

Question 15. Let $a = \dfrac{1 + \sqrt{2009}}{2}$. Find the value of $(a^3 - 503a - 500)^{10}$.

Question 16. In the figure below, ABC is a triangle and D is a point on side BC. Point E is on side AB such that DE is the angle bisector of $\angle ADB$, and point F is on side AC such that DF is the angle bisector of $\angle ADC$. Find the value of $\dfrac{AE}{EB}\dfrac{BD}{DC}\dfrac{CF}{FA}$.

Question 17. Find the value of

$$(\cot 25^0 - 1)(\cot 24^0 - 1)(\cot 23^0 - 1)(\cot 22^0 - 1)(\cot 21^0 - 1)(\cot 20^0 - 1).$$

Question 18. Find the number of 2-element subset $\{a, b\}$ of $\{1, 2, 3, \ldots, 99, 100\}$ such that $ab + a + b$ is a multiple of 7.

Question 19. Let x be real number such that $x^2 - 15x + 1 = 0$. Find the value of $x^4 + \dfrac{1}{x^4}$.

Question 20. In the figure below, ABC is a triangle with $AB = 10$ cm, $BC = 40$ cm. Points D and E lie on side AC and point F on side BC such that EF is parallel to AB and DF is parallel to EB. Given that BE is an angle bisector of $\angle ABC$ and that $AD = 13.5$ cm, find the length of CD in cm.

Question 21. Let $S = \{1, 2, 3, \ldots, 64, 65\}$. Determine the number of ordered triples (x, y, z) such that $x, y, z \in S$, $x < z$ and $y < z$.

Question 22. Given that $a_{n+1} = \dfrac{a_{n-1}}{1 + na_{n-1}a_n}$, where $n = 1, 2, 3, \ldots$, and $a_0 = a_1 = 1$, find the value of $\dfrac{1}{a_{199}a_{200}}\cdot$

Question 23. In the figure below, ABC is a triangle with $AB = 5$ cm, $BC = 13$ cm and $AC = 10$ cm. Points P and Q lie on sides AB and AC respectively such that $\dfrac{\text{area of } \Delta APQ}{\text{area of } \Delta ABC} = \dfrac{1}{4}\cdot$ Given that the least posible length of PQ is k cm, find the value of k.

Question 24. If x, y are real numbers such that $x+y+z = 9$ and $xy+yz+zx = 24$, find the largest possible value of z.

Question 25. Find the number of $0-1$ binary sequences formed by six $0'$s and $1'$s such that no three $0'$s are together. For example, 110010100101 is such a sequence but 101011000101 and 110101100001 are not.

Question 26. If $\dfrac{\cos 100^0}{1 - 4\sin 25^0 \cos 25^0 \cos 50^0} = \tan x$, find x.

Question 27. Find the number of positive integers x, where $x \neq 9$, such that

$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right).$$

Question 28. Let n be the positive integer such that

$$\frac{1}{9\sqrt{11} + 11\sqrt{9}} + \frac{1}{11\sqrt{13} + 13\sqrt{11}} + \frac{1}{13\sqrt{15} + 15\sqrt{11}} + \cdots + \frac{1}{n\sqrt{n+2} + (n+2)\sqrt{n}} = \frac{1}{9}\cdot$$

Find the value of n.

Question 29. In the figure below, $ABCD$ is a rectangle, E is the midpoint of AD and F is the midpoint of CE. If the area of triangle BDF is 12 cm^2, find the earea of rectangle $ABCD$ in cm^2.

Question 30. In each of the following 6-digit positive integers: 555555, 555333, 818811, 300388, every digit in the number appears at least twice. Find the number of such 6-digit positive integers.

Question 31. Let x and y be positive integers such that $27x + 35y \leqslant 945$. Find the largest posible value of xy.

Question 32. Determine the coefficient of x^{29} in the expansion $(1+x^5+x^7+x^9)^{16}$.

Question 33. For $n = 1, 2, 3, \ldots$, let $a_n = n^2+100$, and let d_n denote the greatest common divisor of a_n and a_{n+1}. Find the maximun value of d_n as n ranges over all positive integers.

Question 34. Using the digits $1, 2, 3, 4, 5, 6, 7, 8$, we can form 8! (=40320) 8-digit numbers in which the eight digits are all distinct. For $1 \leqslant k \leqslant 40320$, let a_k denote the hth number if these numbers are arranged in increasing order:

$$12345678, 12345687, 12345768, \ldots, 87654321;$$

that is, $a_1 = 12345678$, $a_2 = 12345687$, $\ldots$, $a_{40320} = 87654321$. Find $a_{2009} - a_{2008}$.

Question 35. Let x be a positive integer, and write $a = \lfloor \log_{10} x \rfloor$ and $b = \left\lfloor \log_{10} \frac{100}{x} \right\rfloor$. Here $\lfloor c \rfloor$ denotes the greatest integer less than or equal to c. Find the largest possible value of $2a^2 - 3b^2$.