2009 Mock Olympiad 1: (APMO Mock)
Time: 4 Hours

1. Let ω be a circle with a diameter BC and XY a chord of ω perpendicular to BC. Points P and M are chosen on XY and CY, respectively, such that $CY||PB$ and $CX||MP$. Let K be the intersection of CX and PB. Prove that PB is perpendicular to MK.

2. Let x, y, z be pairwise distinct non-negative real numbers. Prove that

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{xy+yz+zx}.$$

   Determine when equality holds.

3. Given a set of 2009 points in the plane with no three collinear, prove that for every point $P \in S$, there exists an even number of triangles ABC with $A, B, C \in S$ such that P is in the interior of ΔABC.

4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $(m^2+n)^2$ is divisible by $f(m)^2+f(n)$ for all $m, n \in \mathbb{N}$.

5. Given triangle ABC, let X be a point on ray BC beyond C. Prove that the radical axis of the incircle of ΔABX and the incircle of ΔACX passes through a point independent of X.

2009 Mock Olympiad 1: (APMO Mock)
Time: 4 Hours

1. Let ω be a circle with a diameter BC and XY a chord of ω perpendicular to BC. Points P and M are chosen on XY and CY such that $CY || PB$ and $CX || MP$. Let K be the intersection of CX and PB. Prove that PB is perpendicular to MK.

   Solution: We will consider two cases: If XY is closer to or equidistant to B than to C and if XY is closer to C than to B.

   In the former case, let $\theta = \angle KBC$. By parallel lines and symmetry, we get that $\angle KBC = \angle YCB = \angle KCB$. Hence, $BK = KC$ (1). By parallel lines, we have that $\angle BKX = \angle KCM = 2\theta$ (2). Also, $\angle PXK = 90 - \angle YXB = 90 - \angle YCB = 90 - \theta$. Hence, $\angle XPK = 180 - \angle PXK - \angle BKX = 90 - \theta$. Hence, ΔKXP is isosceles. Therefore, $KX = KP = MC$ (3) since $KPMC$ is a parallelogram. By (1), (2) and (3), we conclude that ΔBKX is congruent to ΔKCM. Hence, $90 = \angle BXK = \angle KMC$, as desired. The second case can be handled similarly.

   Source: Iran Math Olympiad 2005

2. Let x, y, z be non-negative real numbers. Prove that

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{xy+yz+zx}.$$

   Determine when equality holds.

   Solution: Let

$$f(x,y,z) = \frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2}.$$

   and

$$g(x,y,z) = \frac{4}{xy+yz+zx}.$$

   Note that $f(x,y,z) = f(x+d, y+d, z+d)$ for all $d \in \mathbb{R}$. Suppose $x \geq y \geq z$. Then $f(x,y,z) = f(x-z, y-z, 0)$. Note $\frac{4}{xy+yz+zx}$ strictly increases as x, y or z strictly decreases. Hence, $g(x,y,z) \leq g(x-z, y-z, 0)$ with equality if and only if $z = 0$. Hence, it suffices to show that $f(a,b,0) \geq g(a,b,0)$ for all $a, b \geq 0$, i.e.

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{(a-b)^2} \geq \frac{4}{ab}.$$

   After some manipulation, we get this is in fact equivalent to

$$\frac{(a^2 - 3ab + b^2)^2}{a^2 b^2 (a-b)^2} \geq 0$$

with equality if and only if $a^2 - 3ab + b^2 = 0$ or $a/b = (3+\sqrt{5})/2$. Hence, equality holds if and only if $(x, y, z) = t(3+\sqrt{5}, 2, 0)$ for any positive real number t or any of this solution's permutations.

Source: Vietnam Team Selection Test 2008

3. Given a set of 2009 points in the plane with no three collinear, prove that for every point $P \in S$, there exists an even number of triangles ABC with $A, B, C \in S$ such that P is in the interior of ΔABC.

   Solution: We will first prove this problem for five points in the plane. If the convex hull of the five points is a pentagon, then each point is in the interior of zero triangles. If the convex hull of the five points is a quadrilateral $ABCD$, each point on the boundary of this quadrilateral is in the interior of zero triangles. Let AC and BD intersect at Q. For the point in the interior P, P is in exactly one of QAB, QBC, QCD, QDA. Without loss of generality, suppose P is in QAB. Then P is in ABC and ABD, and not BCD, CDA. Hence, P is in the interior of exactly two triangles. If the convex hull of the five points is a triangle ABC, let P, Q be its interior points. Then P is in the interior of ABC, and in the interior of exactly one of QAB, QBC, QCA. Hence, P is in the interior of two triangles. Similarly, Q is in the interior of two triangles. This solves the case for five points.

   Now consider $n > 5$ points in the plane where n is odd. Fix a point $P \in S$. Suppose we count the number of triangles ABC such that P is in ABC. For each unordered triple of points (A, B, C) where $A, B, C \neq P$, let $n_{A,B,C} = 1$ if P is in the interior of ABC and $n_{A,B,C} = 0$ otherwise. For each choice of 4-tuple $A, B, C, D \neq P$, we know that $n_{A,B,C} + n_{A,B,D} + n_{A,C,D} + n_{B,C,D}$ is 0 or 2, from our argument in the scenario for five points. By summing this over all 4-tuples, the eventual sum S is even, and each $n_{A,B,C}$ is counted exactly $n-4$ times, which is an odd number. Hence,

$$\sum_{A,B,C \in S} n_{A,B,C} = \frac{S}{n-4},$$

   which is even, which solves the problem.

Source: British Math Olympiad 2006

4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $(m^2+n)^2$ is divisible by $f(m)^2 + f(n)$ for all $m, n \in \mathbb{N}$.

   Solution: Substituting $m = n = 1$ yields 4 is divisible by $f(1)^2 + f(1)$. Since $f(1) \leq 1$, then $f(1) = 1$. Substituting $m = 1$ yields $(n+1)^2$ is divisible by $1 + f(n)$ for all $n \in \mathbb{N}$. Substituting $n = p-1$ where p is a prime number yields p^2 is divisible by $1 + f(p-1)$. Hence, $f(p-1) + 1 = p$ or p^2 for each prime p. If $f(p-1) + 1 = p^2$, then $m = p-1$ and $n = 1$ yields $((p-1)^2 + 1)^2$ is divisible by $(p^2-1)^2 + 1$. Since $p \geq 2$, the former term is smaller than the latter term. Therefore, this divisibility is impossible. Therefore, $f(p-1) = p-1$ for all primes p.

We now have infinitely many integers k such that $f(k) = k$. Let $k \in \mathbb{N}$ such that $f(k) = k$. Then $k^2 + f(n) = f(k)^2 + f(n)$ divides $(k^2 + n)^2 = (k^2 + f(n) + n - f(n))^2$. This implies $k^2 + f(n)$ divides $(n - f(n))^2$ for infinitely many integers k. Hence, $f(n) = n$ for all $n \in \mathbb{N}$, which satisfies the conditions to the problem.

Source: Romanian Team Selection Test 2006

5. Given triangle ABC, let X be a point on ray BC beyond C. Prove that the radical axis of the incircle of ΔABX and the incircle of ΔACX passes through a point independent of X.

Solution: Let ω_1, ω_2 be the incircles of ΔABX, ΔACX respectively. Let ω_1 touch AX, BX at M, N respectively and ω_2 touch AX, BX at R, S respectively. Therefore, the radical axis passes through the midpoint of M and R and the midpoint of N and S. Call these points P, Q respectively. We now use the following lemma.

Lemma: Given a triangle ABC with an incircle, with incentre I and touches sides BC, CA, AB at D, E, F respectively. Let A', B', C' be midpoints of BC, CA, AB respectively. Then $A'B', DF, AI$ are concurrent.

Proof of Lemma: Let Q be the foot of the perpendicular on AI from C and T be the image of the reflection of C through AI. Hence T is on line AB. Since Q is the midpoint of CC', B', A' are midpoints of A, C and B, C respectively, then B', A', T' are collinear. It remains to show that T, D, F are collinear.

Clearly, I, E, C, Q are concyclic since $\angle IEC = \angle IQC = 90^o$. Since $\angle IDC = \angle ITC = 90^o$, then D is also on this circle, i.e. I, E, C, D, Q are concyclic. Now, $\angle IDF = \angle B/2$. Furthermore, $\angle ICT = \angle ACT - \angle ICT = 90 - \angle A/2 - \angle C/2 = \angle B/2$. Since $ICTD$ is cyclic, $\angle ICT = \angle B/2$ and $\angle IDF = \angle B/2$, we conclude T, D, F are collinear. **End Proof of Lemma**.

Let's return to the original problem. Let l be the line passing through the midpoint of AB and AC. Then by this Lemma, l, MN, and the angle bisector of ABX are concurrent, say U. Also, l, RS, and the angle bisector of ACX (which is independent of X) are concurrent, say at V. Since U, V are independent of X, then PQ passes through the midpoint of U and V, which is also independent of X.

Source: IMO Shortlist 2004

2009 Mock Olympiad 2: (APMO Mock)
Time: 4 Hours

1. Let m, n be positive integers such that $gcd(m, n) = 1$, m is even and n is odd. Evaluate the expression
$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\lfloor \frac{km}{n} \rfloor} \left\{ \frac{km}{n} \right\},$$
where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ (called the fractional part of x).

2. Find all positive real numbers $r \leq 1$ such that there exists an infinite sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive real numbers such that for all $n \geq 1$,
$$x_{n+2} = x_{n+1}^r - x_n^r.$$

3. Let ABC be a triangle with circumcentre O such that the circumradius of ABC is equal to the radius of the excircle opposite A. Let M, N, L be the points where the excircle opposite A touches BC, CA, AB, respectively. Prove that O is the orthocentre of MNL.

4. There are 10 cities in the *Fatland*. Two airlines control all of the flights between the cities. Each pair of cities is connected by exactly one flight (in both directions). Prove that one airline can provide two traveling cycles with each cycle passing through an odd number of cities and with no common cities by the two cycles.

5. Find all polynomials with integer coefficients such that for all positive integers a, b, c, $f(a) + f(b) + f(c)$ is divisible by $a + b + c$.

2009 Mock Olympiad 2: (APMO Mock)
Time: 4 Hours

1. Let m, n be positive integers such that $gcd(m,n) = 1$, m is even and n is odd. Evaluate the expression

$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\lfloor \frac{km}{n} \rfloor} \left\{ \frac{km}{n} \right\},$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ (called the fractional part of x).

Solution: Let $km = q_k n + r_k$ where $1 \leq r_k < n$. Note that since $gcd(m,n) = 1$, then none of $r_k = 0$. Note that $q_k = \lfloor km/n \rfloor$ and $r_k = n\{\frac{km}{n}\}$. We make the following two observations;

(a) Since $gcd(m,n) = 1$, then $\{r_1, r_2, \cdots, r_{n-1}\} = \{1, 2, \cdots, n-1\}$

(b) Since m is even, then km is even. Then since n is odd, then $q_k \equiv r_k \bmod 2$.

Hence, since $n - 1$ is even,

$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\lfloor \frac{km}{n} \rfloor} \left\{ \frac{km}{n} \right\} = \frac{1}{2n} + \frac{1}{n}(-1 + 2 - 3 + \cdots + (n-1)) = \frac{1}{2n} + \frac{n-1}{2n} = \frac{1}{2}.$$

Therefore, the expression is equal to $\frac{1}{2}$.

Source: Romanian Team Selection Test 2005

2. Find all positive real numbers $r \leq 1$ such that there exists an infinite sequence $\{x_n\}_{n \in \mathbb{N}}$ of positive real numbers such that for all $n \geq 1$,

$$x_{n+2} = x_{n+1}^r - x_n^r.$$

Solution: We will prove that this not hold for any positive real number $r \leq 1$. Suppose such a sequence exist. Then for all $n \geq 1$, $x_{n+2} = x_{n+1}^r - x_n^r$. Since $x_{n+2} > 0$, then $x_{n+1} > x_n$ for all $n \geq 1$, implying $\{x_n\}_{n \in \mathbb{N}}$ is a strictly increasing sequence. Hence,

$$x_{n+1}^r - x_n^r > x_n^r - x_{n-1}^r$$

which implies

$$\frac{x_{n+1}^r + x_{n-1}^r}{2} > x_n^r$$

Hence, $\{x_n^r\}_{n \in \mathbb{N}}$ is term by term larger than the arithmetic sequence whose first two elements are x_1^r, x_2^r. Since $x_2^r > x_1^r$, $x_m^r > 1$ for a sufficiently large m, which also implies $x_m > 1$. But $x_{m+1} = x_m^r - x_{m-1}^r < x_m^r \leq x_m$, since $x_m > 1$ and $r \leq 1$, contradicting that $\{x_n\}_{n \in \mathbb{N}}$ is strictly increasing. $\square$

Source: Generalization of a problem in the University of Waterloo Big E Competition 1998

3. Let ABC be a triangle with circumcentre O such that the circumradius of ABC is equal to the radius of the excircle opposite A. Let M, N, L be the points where the excircle opposite A touches BC, CA, AB, respectively. Prove that O is the orthocentre of MNL.

Solution: Let's set up notation first. Let ω be the circumcircle of ABC and γ be the excircle opposite A. Let X be the excentre opposite A and D be the point (which is not A) where AX intersect the circumcentre of ΔABC. Note that $OD \perp BC$. Since the radii of ω and γ are equal, then $OD = XM$. But both $OD, MX \perp BC$, this implies $ODXM$ is a parallelogram, implying $OM||DX$, which implies $OM||AX$. Since $AX \perp LN$, then $OM \perp LN$. Hence, O is on the altitude of ΔMLN from M.

Let the internal angle bisector of $\angle ABC$ intersect ω at E. Note that $\angle EBX = 90^o$ and $|EA| = |EC|$. Let T be the point of intersection of BX and ω (which is not B). Then T is diametrically opposite E which implies $|TA| = |TC|$. Let B' be the foot of the perpendicular on AC from T. Then TB' passes through O. Since the radii of ω and γ are equal, then $OT = NX$. But $OT||NX$, since both are perpendicular to AC. Hence, $TONX$ is a parallelogram. Hence, $TX||ON$. But TX is parallel to BX (same line) which is perpendicular to LM. This implies ON is perpendicular to LM. By symmetry, OL is perpendicular to MN. Hence, O is the orthocentre of ΔMNL as desired. $\Box$

Source: Iran Math Olympiad 2005

4. There are 10 cities in the *Fatland.* Two airlines control all of the flights between the cities. Each pair of cities is connected by exactly one flight (in both directions). Prove that one airline can provide two traveling cycles with each cycle passing through an odd number of cities and with no common cities by the two cycles.

Solution: In graph theory notation, we want to show that a complete graph K_{10}, whose edges are coloured one of two colours, contains a monochromatic subgraph consisting of two disjoint odd cycles. We recall two basic facts.

Lemma 1: Given a complete graph K_6 whose edges are red or blue, there exists a monochromatic triangle.

Lemma 2: Given a complete graph K_5 whose edges are red or blue and no monochromatic triangle, then the red edges and the blue edges both induce a five cycle.

Let v_i, $1 \le i \le 10$ be the vertices of K_{10}. By Lemma 1, there exists a monochromatic triangle, say $v_1v_2v_3$. Suppose this is red. Amongst $v_4v_5, \cdots, v_{10}$, then there eexists a monochromatic triangle, say $v_4v_5v_6$. If this triangle is also red, we are done. Hence, suppose $v_4v_5v_6$ is blue. Consider the edges of the form v_iv_j where $1 \leq i \leq 3$ and $4 \leq j \leq 6$. By Pigeonhole Principle, five of these edges are of the same colour, say red. Then some vertex v_j where $4 \leq j \leq 6$ is incident to two of these red edges. Hence, we have a red and a blue triangle that share exactly one common vertex.

Let's relabel and let $v_1v_2v_3$ be the red triangle and $v_3v_4v_5$ be the blue triangle. Amongst $v_6v_7v_8v_9v_{10}$, if there is red or blue triangle, our problem is solved. Otherwise, by Lemma 2, there is a red 5-cycle and a blue 5-cycle, and we are still done. $\square$

Source: 102 Combinatorial Problems by Andreescu and Feng

5. Find all polynomials with integer coefficients such that for all positive integers a, b, c, $f(a) + f(b) + f(c)$ is divisible by $a + b + c$.

Solution: Recall that for all distinct integers m, n, we have $m - n$ divides $f(m) - f(n)$. Therefore, $a + b + c$ divides $f(a) - f(-(b + c))$. Since $a + b + c$ divides $f(a) + f(b) + f(c)$, we conclude that $a+b+c$ divides $f(b)+f(c)+f(-(b+c))$. This holds for all positive integers a, by choosing a to be sufficiently large, we conclude that $f(b) + f(c) = -f(-(b+c))$. Substituting $c = b$ yields $2f(b) = -f(-2b)$. Let $t = deg(f)$ and $s \neq 0$ be the leading coefficient of f. Then comparing leading coefficients of the left and right side yields $2s = -(-2)^t s$. Since $s \neq 0$, then $2 = -(-2)^t$ implying $t = 1$. Therefore, $f(x) = sx + C$ for some constant C. Hence, $a + b + c$ divides $sa + sb + sc + 3C$ for all a, b, c, implying $a + b + c$ divides $3C$ for all positive integers a, b, c. Therefore, $C = 0$ and $f(x) = sx$ for any positive integer s. This is easily verified as a solution. $\square$

Source: Iran Math Olympiad 2007

2009 Mock Olympiad 3: (APMO Mock)
Time: 4 Hours

1. Let $A_1, A_2, \cdots, A_{100}$ be a collection of subsets of $\{1,2,3,4,5,6\}$ such that for all pairwise distinct i, j, k, we have $|A_i \cup A_j \cup A_k| \geq 5$. Find the minimum possible value of

$$\sum_{t=1}^{100} |A_t|.$$

2. Let r, s be fixed rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = f(x + r) + y + s$$

for all $x, y \in \mathbb{Q}$, where $\mathbb{Q}$ denotes the rational numbers.

3. Let ω be a circle with centre O and l be a line that does not intersect ω. Let Q be the foot of the perpendicular from O on l and P be any point on l different from Q. Let l_1, l_2 be the lines tangent to ω passing through P and A, B be the feet of the perpendicular from Q on l_1, l_2, respectively. Prove that AB passes through a point on OQ independent of the choice of P.

4. Find all pairs of positive integers (m, n) such that

$$3^m = 2^m n + 1.$$

5. Let n be a positive integer and $x_1, x_2, \cdots, x_n \in \mathbb{R}$ such that $|x_i| \leq 1$ for each $i \in \{1, 2, \cdots, n\}$ and $x_1 + x_2 + \cdots + x_n = 0$.

   a.) Prove that there exists $k \in \{1, 2, \cdots, n\}$ such that

$$|x_1 + 2x_2 + \cdots + kx_k| \leq \frac{2k+1}{4}.$$

   b.) For $n > 2$, prove that the bound in (a) is the best possible. i.e. there exists $x_1, x_2, \cdots, x_n$ satisfying the initial conditions such that for all $k \in \{1, 2, \cdots, n\}$.

$$|x_1 + 2x_2 + \cdots + kx_k| \geq \frac{2k+1}{4}.$$

2009 Mock Olympiad 3: (APMO Mock)
Time: 4 Hours

1. Let $A_1, A_2, \cdots, A_{100}$ be a collection of subsets of $\{1,2,3,4,5,6\}$ such that for all pairwise distinct i, j, k, we have $|A_i \cup A_j \cup A_k| \geq 5$. Find the minimum possible value of

$$\sum_{t=1}^{100} |A_t|.$$

Solution: The answer is $70 \cdot 5 + 30 \cdot 4 = 470$.

Let S be the desired sum. To minimize S, we may assume that $|A_i| \leq 5$ for all $1 \leq i \leq 100$. Note that there are 15 subsets of size 4 of $\{1,2,3,4,5,6\}$. I claim that there are at most 30 sets from $A_1, \cdots, A_{100}$ of size at most 4. Suppose not. Suppose there are 31 sets from $A_1, \cdots, A_{100}$ that contains four or less elements. Then there exists a subset of size 4 that contains three of these sets. This contradicts $|A_i \cup A_j \cup A_k| \geq 5$ for all pairwise distinct i, j, k. Therefore, at most 30 sets have size at most 4. Therefore, the other 70 sets must have size 5.

To minimize S, I claim that the remaining 30 sets must all have size 4. Without loss of generality, suppose these 30 sets are $A_1, \cdots, A_{30}$. For each $S \subseteq \{1, 2, \cdots, 6\}$ such that $|S| = 4$, let n_S be the number of subsets in $A_1, \cdots, A_{30}$ that is a subset of S. note that $n_S \leq 2$ for each such S. Otherwise, we have three sets whose union is at most four elements, which is impossible. For each $i \in \{0, 1, 2, \cdots, 5\}$, let a_i be the number of sets amongst $A_1, \cdots, A_{30}$ that have size i.

Let $T = \sum_{S:|S|=4} n_S$. Note that $T \leq 30$. Each set of size 0 contributes 15 to T, each set of size 1 contributes $\binom{5}{3} = 10$ to T, each set of size 2 contributes $\binom{4}{2} = 6$ to T, each set of size 3 contributes $\binom{3}{1} = 3$ to T and each set of size 4 contributes 1 to T. Therefore,

$$15a_0 + 10a_1 + 6a_2 + 3a_3 + a_4 \leq 30.$$

We also note

$$a_0 + a_1 + \cdots + a_4 + a_5 = 30.$$

We want to minimize the expression

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5$$

subject to these two conditions.

We have $a_4 = 30 - (a_0 + a_1 + a_2 + a_3 + a_5)$. Substituting this into the first condition yields

$$15a_0 + 10a_2 + 6a_2 + 3a_3 + 30 - a_0 - a_1 - a_2 - a_3 - a_5 \leq 30$$

or equivalently

$$14a_0 + 9a_1 + 5a_2 + 2a_3 \leq a_5.$$

The expression we want to minimize is

$$\begin{aligned}
&= a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 \\
&= a_1 + 2a_2 + 3a_3 + 4(30 - a_0 - a_1 - a_2 - a_3 - a_5) + 5a_5 \\
&\geq 120 - 4a_0 - 3a_1 - 2a_2 - a_3 + 14a_0 + 9a_1 + 5a_2 + 2a_3 \\
&= 120 + 10a_0 + 6a_1 + 3a_2 + 2a_3 \geq 120.
\end{aligned}$$

Therefore, the expression $|A_1| + \cdots + |A_{30}|$ has minimum possible value 120 and is attainable by taking each subset of 4 exactly twice. This satisfies the condition that the union of any three sets has at least five elements.

The explicit example is; each subset of size 4 is equal to exactly two sets from $A_1, \cdots, A_{30}$. The union of any three of these sets does indeed contain five elements. The remaining 70 sets is any set of size 5. The desired sum does indeed have minimum value 470.

Source: British Mathematical Olympiad 2006

2. Let r, s be fixed rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = f(x + r) + y + s$$

for all $x, y \in \mathbb{Q}$, where $\mathbb{Q}$ denotes the rational numbers.

Solution: The answers are $f(x) = x + r + s$ and $f(x) = -x + r - s$.

To verify these are indeed solutions, if $f(x) = x + r + s$. Then $LHS = f(x + f(y)) = x + y + 2(r + s)$ and $RHS = f(x + r) + y + s = x + 2r + s + y + s = x + y + 2(r + s)$. If $f(x) = -x+r-s$, then $LHS = f(x+f(y)) = f(x-y+r-s) = -x+y-r+s+r-s = -x+y$ and $RHS = f(x + r) + y + s = -x - r + r - s + y + s = -x + y$. Hence, these two solutions do satisfy the given condition. Now we prove that these are the only functions.

Let $g(x) = f(x - s) + r$ [1]. Then $f(x) = g(x + s) - r$. Substituting into the equation in the problem yields

$$g(g(y + s) + x + r + s) = g(x + r + s) + y + s.$$

Note that it suffices to show that $g(x) = \pm x$ are the only solutions. Since $x + r + s, y + s$ take on any real value and are independent of each other, let $u = x + r + s, v = y + s$. Then we have

$$g(g(v) + u) = g(u) + v.(*)$$

[1] The motivation to make this definition is that we want to perform a transformation on $f(x) = x + r + s$ and $f(x) = -x + r - s$ to $g(x) = x$ and $g(x) = -x$ respectively. It is easier to solve for g than it is for f.

Hence, $g(g(g(v)+u)) = g(v)+u$. Since $g(v)+u$ take on all real values, we conclude that $g(g(x)) = x$ for all $x \in \mathbb{Q}$. Hence, g is injective and surjective.

Substituting $v = 0$ into (*) yields $g(g(0)+u) = g(u)$. Since g is injective, then $g(0)+u = u$. Hence, $g(0) = 0$. Finally, substituting $u = g(u)$ into (*) yields $g(g(v)+g(u)) = g(g(u))+v = u+v$. Applying g to both sides yield $g(u)+g(v) = g(u+v)$, which is Cauchy's equation. Since we are over the rationals, then $g(x) = kx$ for some $k \in \mathbb{Q}$. Substituting into (*) yields $k(kv+u) = ku+v$ for all $u, v \in \mathbb{Q}$. Therefore, $k^2 = 1$ implying $k = \pm 1$. Therefore, $g(x) = \pm x$, which implies the given desired solutions for f, which we already verified to satisfy the given condition.

Source: Romanian Team Selection Test 2005

3. Let ω be a circle with centre O and l be a line that does not intersect ω. Let Q be the foot of the perpendicular from O on l and P be any point on l different from Q. Let l_1, l_2 be the lines tangent to ω passing through P and A, B be the feet of the perpendicular from Q on l_1, l_2, respectively. Prove that AB passes through a point on OQ independent of the choice of P.

   Solution: Let l_1, l_2 touch ω at X, Y respectively. Since $\angle OXP = \angle OYP = \angle OQP = 90^o$, then P, Q, Y, O, X are concyclic. Let C be the foot of the perpendicular on XY from Q. Let XY intersect OQ at M. By the Simson Line Theorem, A, B, C are collinear. Let this line intersect OQ at N. We need to show that N is independent of P.

   Note that X, Y are on opposite sides of line OQ. Without loss of generality, suppose P is on the same side of OQ as X.

   Let $\angle YPQ = \alpha$, $\angle XPY = 2\theta$. Therefore, $\angle OXY = \angle OYX = \theta$. Since P, Q, Y, O, X are concyclic, then $\angle YOQ = \alpha$. Since $ABQP$ is cyclic, $\angle BAQ = \angle BPQ = \alpha$. Hence, $\angle XAC = 90 - \alpha$. Since $\angle OXY = \theta$, $\angle AXC = 90 - \theta$. Therefore, $\angle ACX = \theta + \alpha$. However, $\angle CMN = \angle MOY + \angle MYO = \theta + \alpha$ and $\angle MCN = \angle ACX = \theta + \alpha$. Therefore, $CM = CN$. But since $\angle MCQ = 90^o$, then N is the midpoint of MQ. Since Q is a fixed point (independent of P), it suffices to show that M is independent of P.

   Note that $\angle OQY = \angle OPY = \theta = \angle OYM$. Therefore, $OM \cdot OQ = OY^2$ by Power of a Point. Since OQ, OY are independent of P, so is OM. (Note that the line XMY does separate O and Q into two separate planes, so this argument does work.) This proves the problem. $\square$

   Source: IMO Shortlist 1994

4. Find all pairs of positive integers (m, n) such that

$$3^m = 2^m n + 1.$$

Solution: The only solutions are $(1,1),(2,2)$ and $(4,5)$.

Substituting $m=1$ yields the first solution. Suppose now $m \geq 2$. Taking mod 4 on both sides, we see that m is even. Therefore, let $m = 2^r(2s+1)$ where $r,s \in \mathbb{Z}$, $r \geq 1$, $s \geq 0$. Rearranging terms and factoring yield

$$2^m n = (3^m - 1) = (3^{2s+1}-1)(3^{2s+1}+1)(3^{2\cdot(2s+1)}+1)\cdots(3^{2^{r-1}(2s+1)}+1)$$

We now see how many powers of 2 divide both side of the equation.

By taking modulo 4, we see that $3^{2s+1}-1 \equiv (-1)^{2s+1}-1 = -2 \bmod 4$. Therefore $3^{2s+1}-1$ is divisible by 2, but not by 4.

By taking modulo 8, we see that $3^{2s+1}+1 \equiv 3\cdot 9^s + 1 \equiv 3+1 \equiv 4 \bmod 8$. Hence, $3^{2s+1}+1$ is divisible by 4 but not by 8.

By taking modulo 4, we see that $3^{2t(2s+1)}+1 \equiv (-1)^{2t(2s+1)}+1 \equiv 2 \bmod 4$. Hence, $3^{2^t(2s+1)}+1$ is divisible by 2 but not by 4.

Therefore, in (*), the highest power of 2 that divides the right-hand-side is $r+2$. i.e. $2^{r+2}||RHS$. But $2^{2^r(2s+1)}$ divides LHS. Therefore, $r+2 \geq 2^r(2s+1)$. If $r=1$, then $s=0$. This yields the solution $(2,2)$ If $r=2$, then $s=0$. This yields the solution $(4,5)$. If $r \geq 3$, then $r+2 < 2^r(2s+1)$. Hence, there are no more solutions. $\square$

Source: Romanian Team Selection Test 2005

5. Let n be a positive integer and $x_1, x_2, \cdots, x_n \in \mathbb{R}$ such that $|x_i| \leq 1$ for all $i \in \{1,2,\cdots,n\}$ and $x_1 + x_2 + \cdots + x_n = 0$.

   a.) Prove that there exists $k \in \{1,2,\cdots,n\}$ such that

$$|x_1 + 2x_2 + \cdots + kx_k| \leq \frac{2k+1}{4}.$$

   b.) For $n > 2$, prove that the bound in (a) is the best possible. i.e. there exists $x_1, x_2, \cdots, x_n$ satisfying the initial conditions such that for all $k \in \{1,2,\cdots,n\}$.

$$|x_1 + 2x_2 + \cdots + kx_k| \geq \frac{2k+1}{4}.$$

Solution: (a) Let $S_t = x_1 + 2x_2 + \cdots + tx_t$. Since S_t remains unchange by changing the signs of all the x_i's, we may assume that $x_1 \geq 0$. Therefore,

$$x_t = \frac{S_t - S_{t-1}}{t},$$

where we use the notation $S_0 = 0$. Therefore,

$$0 = x_1 + x_2 + \cdots + x_n = \sum_{t=1}^{n} \frac{S_t - S_{t-1}}{t} = \frac{S_n}{n} + \sum_{t=1}^{n-1} (\frac{1}{t-1} - \frac{1}{t}) S_t = \frac{S_n}{n} + \sum_{t=1}^{n-1} \frac{1}{t(t+1)} S_t.$$

Therefore, at least one S_k is negative. Let k be the smallest index such that $S_k < 0$.

However, $k \geq |kx_k| = |S_k - S_{k-1}| = S_{k-1} - S_k = |S_{k-1}| + |S_k|$. But suppose $|S_{k-1}| > \frac{2k-1}{4}$ and $|S_k| > \frac{2k+1}{4}$. Then $|S_{k-1}| + |S_k| > k$. Contradiction. Therefore, $|S_k| < \frac{2k+1}{4}$

(b) If n is odd, let $(x_1, x_2, \cdots, x_n) = (3/4, 1/4, -1, 1, \cdots, -1)$. The sum is zero and for each k, $S_k \geq \frac{2k+1}{4}$.

If n is even, let $(x_1, x_2, \cdots, x_n) = (1, 1/8, -1, -1/8, 1, -1, \cdots, 1, -1)$. $\square$

Source: Romanian Team Selection Test 2006

2009 Mock Olympiad 4: (APMO Mock)
Time: 4 Hours

1. Let n, k be positive even integers. A survey was done on n people where on each of k days, each person was asked whether he/she was happy on that day and answered either "yes" or "no". It turned out that on any two distinct days, exactly half of the people gave different answers on the two days. Prove that there were at most $n - \frac{n}{k}$ people who answered "yes" the same number of times he/she answered "no" over the k days.

2. Given a triangle ABC with an interior point P, let A_1 be the intersection of AP with the circumcircle of ΔPBC which is not P. Define B_1, C_1 analogously. Prove that

$$\left(1 + 2 \cdot \frac{|PA|}{|PA_1|}\right)\left(1 + 2 \cdot \frac{|PB|}{|PB_1|}\right)\left(1 + 2 \cdot \frac{|PC|}{|PC_1|}\right) \geq 8.$$

3. Find all prime numbers p such that $p = m^2 + n^2$ and p divides $m^3 + n^3 - 4$ for some positive integers m, n .

4. Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

Let M be the midpoint of CD. Prove that AM, BD, CE are concurrent.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y) + f(x)) = 2f(x) + xy$$

for all $x, y \in \mathbb{R}$.

2009 Mock Olympiad 4: (APMO Mock)
Time: 4 Hours

1. Let n, k be positive even integers. A survey was done on n people where on each of k days, each person was asked whether he/she was happy on that day and answered either "yes" or "no". It turned out that on any two distinct days, exactly half of the people gave different answers on the two days. Prove that there were at most $n - \frac{n}{k}$ people who answered "yes" the same number of times he/she answered "no" over the k days.

   Solution: Let $P_1, \cdots, P_n$ be the people and $D_1, D_2, \cdots, D_k$ be the k days. We count the number of triples (P_i, D_j, D_k) (where $j < k$) where person P_i gave different answers on D_j and D_k. Since for every two days, exactly half of the people gave different answers on the two days, the number of such triples is

$$\frac{n}{2} \cdot \binom{k}{2} = \frac{nk(k-1)}{4}.$$

   Each person who answered "yes" the same number of times he answered "no" is in $(k/2)(k/2) = k^2/4$ such triples. Hence, the number of people who answered "yes" the same number of times he answered "no" is at most

$$\frac{\frac{nk(k-1)}{4}}{\frac{k^2}{4}} = \frac{n(k-1)}{k} = n - \frac{n}{k}$$

   as desired. $\square$

   Source: Iran Mathematical Olympiad 2006

2. Given a triangle ABC with an interior point P, let A_1 be the intersection of AP with the circumcircle of ΔPBC which is not P. Define B_1, C_1 analogously. Prove that

$$\left(1 + 2 \cdot \frac{|PA|}{|PA_1|}\right)\left(1 + 2 \cdot \frac{|PB|}{|PB_1|}\right)\left(1 + 2 \cdot \frac{|PC|}{|PC_1|}\right) \geq 8.$$

   Solution: I am sure there is a bash solution. But why bash when you can invert about P. For a point X, let X' denote its image of the inversion.

   Then note that B', C', A_1' are collinear since A_1, B, C, P are conyclic. Similarly, A', B', C_1' are collinear and C', A', B_1' are collinear. Consider triangle $\Delta A'B'C'$. The sides $B'C', C'A', A'B'$ contain points A_1', B_1', C_1', respectively. Furthermore, $A'A_1', B'B_1, C'C_1$ are concurrent. Finally, note that

$$\frac{|PA|}{|PA_1|} = \frac{|PA_1'|}{|PA'|} = \frac{[PB'C']}{[A'B'C'] - [PB'C']}$$

   where $[XYZ]$ denotes area of triangle XYZ. Similarly,

$$\frac{|PB|}{|PB_1|} = \frac{[PC'A']}{[A'B'C'] - [PC'A']}, \frac{|PC|}{|PC_1|} = \frac{[PA'B']}{[A'B'C'] - [PA'B']}.$$

Let $x = [PB'C'], y = [PC'A'], z = [PA'B']$. Then $x + y + z = [A'B'C']$. Then the problem becomes equivalent to

$$\left(1 + 2 \cdot \frac{x}{y+z}\right)\left(1 + 2 \cdot \frac{y}{z+x}\right)\left(1 + 2 \cdot \frac{z}{x+y}\right) \geq 8.$$

Simplifying this reduces the problem to proving

$$2(x^3 + y^3 + z^3) + 7\sum_{sym} x^2y + 16xyz \geq 8\sum_{sym} x^2y + 16xyz$$

or equivalently

$$2(x^3 + y^3 + z^3) \geq x^2y + xy^2 + y^2z + yz^2 + z^2x + xz^2$$

or equivalently

$$(x + y)(x - y)^2 + (y + z)(y - z)^2 + (z + x)(z - x)^2 \geq 0$$

which is true. $\square$

Source: China Team Selection Test 2008

3. Find all prime numbers p such that $p = m^2 + n^2$ and p divides $m^3 + n^3 - 4$ for some positive integers m, n.

Solution: The answers are $p = 2$ and $p = 5$.

Computing modulo p, we have $m^3 + n^3 - 4 = (m + n)(m^2 - mn + n^2) - 4$. Hence, p divides $-mn(m + n) - 4$ which divides $2mn(m + n) + 8$. Since $(m + n)^2 \equiv 2mn \bmod p$, then p divides $(m + n)^3 + 8 = (m + n + 2)((m + n)^2 - 2(m + n) + 4)$. Therefore, $p \mid m + n - 2$ or $p \mid (m + n)^2 + 2(m + n) + 4$.

If $p \mid m + n + 2$, then either $m^2 + n^2 \leq m + n + 2$ or $m + n + 2 = 0$. The latter case is impossible. In the former case, the expression is equivalent to $(2m - 1)^2 + (2n - 1)^2 \leq 10$. Hence, $m, n \leq 2$. If $m = 1, n = 1$, then $p = 2$, which is prime. Then $m^3 + n^3 - 4 = -2$ and therefore $2 \mid m^3 + n^3 - 4$. If $m = n = 2$, then $m^2 + n^2 = 8$, which is not prime. If $m = 1, n = 2$, then $p = 1^2 + 2^2 = 5$ which is prime. $m^3 + n^3 - 4 = 5$. Therefore, $p \mid m^3 + n^3 - 4$. Hence, $p = 2, 5$ works.

We may assume now that p is an odd prime. If $p \mid (m + n)^2 - 2(m + n) + 4$, then since $p = m^2 + n^2$, we have $p \mid 2mn - 2m - 2n + 4$. Since p is odd, $p \mid mn - m - n + 2$. Therefore, $m^2 + n^2 \leq mn - m - n + 2$ or $mn - m - n + 2 = 0$. The latter is impossible since $mn - m - n + 2 = (m - 1)(n - 1) + 1 > 0$. The former case is equivalent to $(m - n)^2 + (m + 1)^2 + (n + 1)^2 \leq 6$. Therefore, $m, n \leq 1$, whose case has been accounted for.

Hence, the only solutions are $p = 2, 5$. $\square$

Source: Iran Mathematical Olympiad 2004

4. Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

Let M be the midpoint of CD. Prove that AM, BD, CE are concurrent.

Solution: Clearly, $\Delta ABC \sim \Delta ACD \sim \Delta ADE$. Therefore,

$$\frac{AB}{BC} = \frac{AC}{CD} = \frac{AD}{DE}(1), \text{ and } \frac{BC}{AC} = \frac{CD}{AD} = \frac{DE}{AE}(2)$$

Therefore, $AB/AD = BC/DE = AC/AE$. Since $\angle BAD = \angle CAE$, $\Delta ABD \sim \Delta ACE$.

Then let $P = BD \cap CE$. Then $\angle BDA = \angle CEA \Rightarrow \angle PDA = \angle PEA$. Hence, $APDE$ is cyclic. Let the circle passing through these four points be ω_1. Hence, $\angle DAE = \angle DPE = \angle BPC = \angle BAC$. Hence, $ABCP$ is also cyclic. Let the circle passing through these four points be ω_2. Then AP is the radical axis of these two circles. To prove AP intersects CD at M, it suffices to show that ω_1 and ω_2 are tangent to CD, as M would lie on the radical axis of the two circles and have the same power with respect to both circles.

Since $\angle ABD = \angle ACE$ and $\angle ABC = \angle ACD$, then $\angle PBC = \angle PCD$. Hence, ω_1 is tangent to CD. Since $\angle ADC = \angle AED$ and $\angle AEC = \angle ADB$, then $\angle DEP = \angle PDC$. Hence, ω_2 is tangent to CD. Hence, the radical axis, AP intersects CD at M. Therefore, AM, BD, CE are concurrent as desired. $\square$

Source: IMO Shortlist 2006

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y) + f(x)) = 2f(x) + xy$$

for all $x, y \in \mathbb{R}$.

Solution: The answer is $f(x) = x + 1$.

We first prove that f is injective; suppose $f(a) = f(b)$. Then substituting $y = a$, and $y = b$ separately will yield that $2f(x) + xa = 2f(x) + xb$ for all $x \in \mathbb{R}$. Hence, $a = b$. Therefore, f is injective.

We next prove f is surjective, if we fix a non-zero x, the the right-hand side of the equation can take on any real value as we vary y. This right-hand side is also in the range $f(xf(y)+f(x))$. Therefore, f is surjective.

We next prove that $f(0) = 1$ and $f(-1) = 0$. Let $f(0) = a$ and $f(b) = 0$. Then substituting $x = b, y = 0$ into the original equation yields $f(ba) = 0$. Since f is injective and $f(b) = 0$, $ba = b$. If $b = 0$, then $f(0) = 0$. Substituting $y = 0$ into the original equation yields $f(f(x)) = 2f(x)$. Since f is surjective, this yields $f(t) = 2t$ for all $t \in \mathbb{R}$. However, by checking, we see that this is *not* a solution. Therefore, $a = 1$, which implies $f(0) = 1$. Now, substituting $x = y = b$ into the original equation yields $f(f(b)) = b^2$. Hence, $f(0) = b^2$. Since $f(0) = a = 1$, then $b^2 = 1$. If $b = 1$, then $f(1) = 0$. Substituting $y = b$ fields $f(f(x)) = 2f(x) + x$. Substituting $x = 0$ yields $0 = f(f(0)) = 2f(0) + 0 = 2$, contradiction. Therefore, $b \neq 1$. Since $b^2 = 1$, this implies $b = -1$. Therefore, $f(0) = 1$ and $f(-1) = 0$.

Substituting $y = -1$ yields

$$f(f(x)) = 2f(x) - x, \forall x \in \mathbb{R}.(1)$$

Substituting $x = 1$, followed by $y = x$ yields

$$f(f(x) + 2) = 4 + x, \forall x \in \mathbb{R}.(2)$$

Adding (1) and (2) yields $f(f(x) + 2) + f(f(x)) = 2f(x) + 4$. Since f is surjective, the substitution $x = f(x)$ yields

$$f(x + 2) + f(x) = 2x + 4.(3)$$

Substituting $x = x + 2$ into (3) yields

$$f(x + 4) + f(x + 2) = 2x + 8.(4)$$

Next, we want to expression $f(x + 4) + f(x)$ in terms of x. Applying f to both sides of (2) and applying (1) yields

$$f(f(f(x)+2)) = f(4+x) \Rightarrow 2f(f(x)+2)-(f(x)+2) = f(x+4) \Rightarrow 2(x+4)-f(x)-2 = f(x+4)$$

$$f(x + 4) + f(x) = 2x + 6.(5)$$

Adding (3), (4), (5) and dividing by 2 yields

$$f(x + 4) + f(x + 2) + f(x) = 3x + 9.(6)$$

Subtracting (4) from (6) yields $f(x) = x + 1$. Finally, we verify that this solution works. If $f(x) = x + 1$, we get

$$f(xf(y) + f(x)) = f(x(y + 1) + x + 1) = xy + x + x + 1 + 1 = xy + 2x + 2$$

and

$$2f(x) + xy = 2(x + 1) + xy = xy + 2x + 2.$$

Therefore, $f(x) = x + 1$ does satisfy the original equation. We are done. $\square$

Source: Brazilian Mathematical Olympiad 2006

2009 Mock Olympiad 5: (APMO Mock)
Time: 4 Hours

1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $a_1 = 1/2$ and

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}$$

for all $n \in \mathbb{N}$. Prove that $a_1 + a_2 + \cdots + a_N < 1$ for all positive integers N.

2. Given triangle ABC with $|AB| < |AC|$, let P be on side AC such that $|CP| = |AB|$ and Q be on ray BA such that $|BQ| = |AC|$. Let R be the intersection of PQ and the perpendicular bisector of BC. Prove that

$$\angle BAC + \angle BRC = 180^o.$$

3. Prove that for any non-negative integer n, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k} 4^{n-k} 3^k$$

is the sum of two consecutive perfect squares.

4. Let $x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$

5. A finite set of (pairwise distinct) positive integers is said to be *divisible-friendly* if every element in the set divides the sum of all of the elements in the set. Prove that every finite set of positive integers is the subset of a divisible-friendly set of positive integers.

2009 Mock Olympiad 5: (APMO Mock)
Time: 4 Hours

1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $a_1 = 1/2$ and

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}$$

for all $n \in \mathbb{N}$. Prove that $a_1 + a_2 + \cdots + a_N < 1$ for all positive integers N.

Solution: For each $n \in \mathbb{N}$, let $b_n = \frac{1}{a_n}$. (Since $a_n = 0$ if and only if $a_{n-1} = 0$, and $a_1 = 1/2$, and $a_n^2 - a_n + 1 > (a_n - 1/2)^2 \geq 0$, each a_n is positive.)

By substituting into the original equation we get that

$$\frac{1}{b_{n+1}} = \frac{\frac{1}{b_n^2}}{\frac{1}{b_n^2} - \frac{1}{b_n} + 1} = \frac{1}{b_n^2 - b_n + 1}$$

or equivalently,

$$b_{n+1} = b_n^2 - b_n + 1.$$

Since $b_1 = 2$, we get easily that $b_2 = 3, b_3 = 7, b_4 = 43$. We will prove by induction that $b_{n+1} = 1 + b_1 b_2 \cdots b_n$ for all positive integers n. This holds clearly for $n = 1$ since $b_2 = 3$ and $1 + b_1 = 3$. Suppose this holds for $n = k$ for some positive integer k. Then $b_{k+1} = b_k^2 - b_k + 1 = b_k(b_k - 1) + 1 = b_k(b_{k-1}b_{k-2}\cdots b_1)$ as desired. This proves that $b_{n+1} = 1 + b_1 b_2 \cdots b_n$ for all positive integers n.

I finally claim that $a_1 + a_2 + \cdots + a_N = 1 - a_1 a_2 \cdots a_N$ for all positive integers N. This holds for $N = 1$ since $a_1 = 1/2$ and $1 - a_1 = 1/2$. Suppose this holds for $N = k$. Then

$$a_1 + a_2 + \cdots + a_{k+1} = 1 - a_1 a_2 \cdots a_k + a_{k+1} = 1 - \frac{1}{b_1 b_2 \cdots b_k} + \frac{1}{b_{k+1}} = 1 - \frac{b_{k+1} - b_1 b_2 \cdots b_k}{b_1 b_2 \cdots b_{k+1}}$$

$$= 1 - \frac{1}{b_1 b_2 \cdots b_{k+1}} = 1 - a_1 a_2 \cdots a_{k+1}$$

which completes the induction proof. Hence, $a_1 + a_2 + \cdots + a_N = 1 - a_1 a_2 \cdots a_N < 1$ as desired. $\square$

Source: Romanian Math Olympiad 2003

2. Given triangle ABC with $|AB| < |AC|$, let P be on side AC such that $|CP| = |AB|$ and Q be on ray BA such that $|BQ| = |AC|$. Let R be the intersection of PQ and the perpendicular bisector of BC. Prove that

$$\angle BAC + \angle BRC = 180^o.$$

Solution: Note that $|AQ| = |BQ| - |AB| = |AC| - |CP| = |AP|$. Therefore, $|AQ| = |AP|$. Since $\angle QAP = 180 - \angle A, \angle AQP = \angle APQ = \angle A/2$.

Let A' be external to triangle ABC such that $ACA'B$ is a parallelogram. Then note $\Delta ABC \cong \Delta A'CB$. Note also that $\angle ABA' = 180 - \angle A$. Consider $\Delta BQA'$. Since $\angle QBA = 180 - \angle A$ and $\angle BQR = \angle A/2$, then the angle bisector of $\angle BA'C$ passes through R and Q. Since R is on the perpendicular bisector of BC and is on the angle bisector of $\angle BA'C$, then R is on the circumcircle of $\Delta A'BC$. Furthermore, R is external to $\Delta A'BC$. Therefore,

$$\angle BAC + \angle BRC = \angle BA'C + \angle BRC = 180^o$$

as desired. $\square$

Source: British Mathematical Olympiad 2006

3. Prove that for any non-negative integer n, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k} 4^{n-k} 3^k$$

is the sum of two consecutive perfect squares.

Solution: Let S_n be the integer described in the question. It is easy to see that $S_0 = 1$ and $S_1 = 13$. Then

$$\begin{aligned} S_n = \sum_{k=0}^{n} \binom{2n+1}{2k} 4^{n-k} 3^k = \sum_{k=0}^{n} \binom{2n+1}{2k} 2^{2n-2k} \sqrt{3}^{2k} = \frac{1}{2} \sum_{k=0}^{n} \binom{2n+1}{2k} 2^{(2n+1)-k} \sqrt{3}^{2k} \\ = \frac{1}{4}((2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1}) \end{aligned}$$

by the Binomial Theorem.

We prove the following lemma; a positive integer N is the sum of two consecutive perfect squares if and only if $2N - 1$ is a perfect square. This is easy to prove. We leave it to the reader.

Therefore, it suffices to show that $2S_n - 1$ is a perfect square. Note that

$$\begin{aligned} 2S_n - 1 &= \frac{2+\sqrt{3}}{2}(2+\sqrt{3})^{2n} + \frac{2-\sqrt{3}}{2}(2-\sqrt{3})^{2n} - 1 \\ &= \frac{(1+\sqrt{3})^2}{4}(2+\sqrt{3})^{2n} + \frac{(1-\sqrt{3})^2}{4}(2-\sqrt{3})^{2n} - 1 \\ &= \left(\frac{1+\sqrt{3}}{2}(2+\sqrt{3})^n + \frac{1-\sqrt{3}}{2}(2-\sqrt{3})^n\right)^2 \end{aligned}$$

I claim that this expression is indeed a perfect square. Let $T_n = \frac{1+\sqrt{3}}{2}(2+\sqrt{3})^n + \frac{1-\sqrt{3}}{2}(2-\sqrt{3})^n$. Note that $T_0 = 1$ and $T_1 = 5$. Since $2+\sqrt{3}, 2-\sqrt{3}$ are roots of $x^2 - 4x + 1$,

$$T_{n+2} = 4T_{n+1} - T_n,$$

which implies the T_n's are integers. Therefore, T_n are integers which implies $2S_n - 1$ are indeed integral perfect squares. By the Lemma, S_n is the sum of two consecutive perfect squares. □

Source: Romanian Mathematical Olympiad 1999

4. Let $x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$

Solution: Suppose $\sum\limits_{i=1}^{n} \frac{1}{n-1+x_i} > 1$. Then for each $j \in \{1, 2, \cdots, n\}$,

$$\frac{1}{n-1+x_j} > 1 - \sum_{i \neq j} \frac{1}{n-1+x_j} = \sum_{i \neq j} \frac{1}{n-1} - \frac{1}{n-1+x_i} = \sum_{i \neq j} \frac{x_i}{(n-1)(n-1+x_i)}$$
$$\geq \prod_{i \neq j} \sqrt[n-1]{\frac{x_i}{n-1+x_i}}.$$

by the AM-GM inequality. Therefore,

$$\prod_{j=1}^{n} \frac{1}{n-1+x_j} > \prod_{j=1}^{n} \prod_{i \neq j} \sqrt[n-1]{\frac{x_i}{n-1+x_i}} = \prod_{j=1}^{n} \frac{x_j}{n-1+x_j} = \prod_{j=1}^{n} \frac{1}{n-1+x_j},$$

contradiction. Hence, the equality does hold. □

Source: Romanian Mathematical Olympiad 1999

5. A finite set of (pairwise distinct) positive integers is said to be *divisible-friendly* if every element in the set divides the sum of all of the elements in the set. Prove that every finite set of positive integers is the subset of a divisible-friendly set of positive integers.

Solution 1: It suffices to show that for every positive integer n, there exists a divisible-friendly set that contains $\{1, 2, \cdots, n\}$. The sum of these elements is $n(n+1)/2$. We add $n(n+1)/2$ to the set. Now, the sum of the elements is $n(n+1)$. My goal is to create a divisibly-friendly set containing these $n+1$ elements whose sum is $(n+1)!$. The sum of the remaining elements must be $(n+1)! - n(n+1)$. However, by telescoping sums, we have

$$(n+1)! - n(n+1) = \sum_{j=1}^{n-1} ((n+1)n(n-1)\cdots j - (n+1)n(n-1)\cdots(j+1))$$

$$= \sum_{j=1}^{n-1} (n+1)n(n-1)\cdots(j+1)(j-1).$$

Hence, if I add the terms $(j-1)(j+1)(j+2)\cdots n(n+1)$ for $j \geq 2$, then the sum of all elements is $n!$ and clearly, every term in the set divides $(n+1)!$ and the terms are pairwise distinct. We have created our divisible-friendly set. $\square$

Solution 2: Let S be a finite set of integers. If $|S| = 1$, then clearly S is divisible-friendly. Now suppose $|S| > 1$. Let N be the sum of the elements in S. Let $x \in S$ such that $x \nmid N$. I will create a set S' that contains S, x divides the sum of the elements in S', if $y \in S$ divides N, then y also divides the sum of the elements in S' and for all $y \in S' \backslash S$, y divides the sum of the elements in S'. Repeated application of this will create a divisible-friendly set, since the number of elements in the sets we create that does not divide the set's sum, strictly decreases. This will finish the problem.

Let $x \in S$ such that x does not divide N. We write $x = 2^m \cdot n$, where m is a non-negative integer and n is a positive odd integer. (Clearly $x \neq 1$.) If $n = 1$, then add to S the elements $N, 2N, \cdots, 2^{m-1}N$. The sum of S is now $2^m N$. Clearly, if $y \mid N$, then $y \mid 2^m N$. Also, $2^i N \mid 2^m N$ for all $0 \leq i < m$. Finally, $x = 2^m \mid 2^m N$. In the case where x is a power of 2, we are done.

Now suppose $x = 2^m \cdot n$ where n is an odd positive integer greater than one. Again, we add $N, 2N, \cdots, 2^{m-1}N$. The sum so far is $2^m N$. Of course, x still may not divide $2^m N$. Let t be a positive integer such that $N \mid 2^t - 1$. (For example, choose $t = \varphi(n)$ by Euler's Theorem.) We further add the following elements to the set;

$$2 \cdot 2^m N, 2^2 \cdot 2^m N, \cdots, 2^{t-1} \cdot 2^m N$$

The new elements have sum $(2^t - 2)2^m N$. Hence, the sum of the sets of elements is $(2^t - 1)2^m N$. Therefore, n divides this sum and therefore x divides this sum. But $2^i \cdot 2^m N$ may not. Damn. Let's add even more elements. Let's add

$$(2^t - 1) \cdot 2^m N, 2(2^t - 1) \cdot 2^m N, 2^2(2^t - 1) \cdot 2^m N, \cdots, 2^{t-1}(2^t - 1) \cdot 2^m N.$$

The sum of these new elements is $(2^t - 1)(2^t - 1) \cdot 2^m N$ and therefore the sum of all of the elements is $2^t(2^t - 1)2^m N$. Now, $2^i 2^m N$ divides this new sum for all $0 \leq i \leq t-1$ and $2^i(2^t - 1) \cdot 2^m N$ also divides this new sum. And we are finally done. Our set works. $\square$

Source: United States of America Mathematical Olympiad Proposal 1997

2009 Mock Olympiad 6: (APMO Mock)
Time: 4 Hours

1. Let n, k be positive integers. In a certain library, there are n shelves, each holding at least one book. k new shelves are acquired and the books are arranged on the $n+k$ shelves, again with at least one book on each shelf. A book is said to be *special* if it is in a shelf with fewer books in the new arrangement than it was in the original arrangement. Prove that there are at least $k+1$ special books in the rearranged library.

2. Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$. Prove that
$$a^2 + b^2 - c^2 > 6(c-a)(c-b).$$

3. Given triangle ABC, whose incircle touches BC, CA, AB at D, E, F, respectively, let AD intersect the incircle again at P. Let Q be the intersection of EF and the line passing through P perpendicular to AD. Let AQ intersect DE, DF at X, Y respectively. Prove that A is the midpoint of XY.

4. For a positive integer a , let S_a be the set of primes p for which there exists an odd integer b such that p divides $(2^{2^a})^b - 1$. Prove that for every a, there exist infinitely many primes that are not contained in S_a.

5. Find the largest positive integer k such that for any ten points in the plane with the property that every subset of five points contains at least four concyclic points, there exist k concyclic points.

2009 Mock Olympiad 6: (APMO Mock)
Time: 4 Hours

1. Let n, k be positive integers. In a certain library, there are n shelves, each holding at least one book. k new shelves are acquired and the books are arranged on the $n+k$ shelves, again with at least one book on each shelf. A book is said to be *special* if it is in a shelf with fewer books in the new arrangement than it was in the original arrangement. Prove that there are at least $k+1$ special books in the rearranged library.

   Solution: Let $B_1, B_2, \cdots, B_m$ be the books. For each $i \in \{1, 2, \cdots, m\}$, let a_i, b_i be the number of books on shelf containing B_i before and after the acquired shelves arrive, respectively. Then note that
$$\sum_{i=1}^{n} \frac{1}{a_i} = n \text{ and } \sum_{i=1}^{n} \frac{1}{b_i} = n + k.$$
   Note that a book B_i is special if and only if $b_i < a_i$. Since the difference between these two expressions is k, and two terms of the form $\frac{1}{t}$, where t is a positive integer, differ by strictly less than 1, $b_i < a_i$ for at least $k+1$ values of i. Therefore, there are at least $k+1$ special books. $\square$

   Source: Australian Mathematical Olympiad 1990

2. Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$. Prove that
$$a^2 + b^2 - c^2 > 6(c-a)(c-b).$$

   Solution: Since the equation and the initial condition are homogeneous, we may assume that $c = 1$. Therefore, it suffices to prove the inequality
$$a^2 + b^2 - 1 > 6(1-a)(1-b)$$
   where $a^3 + b^3 = 1$. By expanding, this inequality is equivalent to
$$a^2 + b^2 + 6(a+b) - 6ab - 7 > 0.$$
   Let $a + b = u, ab = v$. Since $a^3 + b^3 = 1$, we have $(a+b)(a^2 - ab + b^2) = 1$. This implies $u(u^2 - 3v) = 1$. Therefore, $v = \frac{u^3-1}{3u}$.

   Hence our inequality becomes equivalent to
$$u^2 - 2v + 6u - 6v - 7 > 0 (\Leftrightarrow) u^2 + 6u - 8\frac{u^3-1}{3u} - 7 > 0 (\Leftrightarrow) 5u^3 - 18u^2 + 21u - 8 < 0.$$
   This factors as
$$(u-1)^2(5u-8) < 0.$$

This inequality is true if and only if $u < 8/5$, i.e. $a+b < 8/5$. Since $a^3+b^3 = 1$ and $(a+b)^3 = a^3+b^3+3ab(a+b) \le a^3+b^3+3(a^3+b^3) = 4$, we conclude that $a+b \le \sqrt[3]{4} < \sqrt[3]{512/125} = 8/5$. This proves the inequality.

Source: Indian Mathematical Olympiad 2009

3. Given triangle ABC, whose incircle touches BC, CA, AB at D, E, F, respectively, let AD intersect the incircle again at P. Let Q be the intersection of EF and the line passing through P perpendicular to AD. Let AQ intersect DE, DF at X, Y respectively. Prove that A is the midpoint of XY.

Solution: Let M be the midpoint of EF and I be the centre of ΔABC. Note that A, M, I are collinear. First, we note that $|AE|^2 = |AF|^2 = |AI| \cdot |AM|$. (You can prove this latter fact in one of many ways.) Therefore, $|AM| \cdot |AI| = |AF|^2 = |AP| \cdot |AD|$ by power of a point. Therefore, $PMID$ is cyclic. Therefore, $\angle AMP = \angle ADI$ since A, M, I are collinear and $|AM| \cdot |AI| = |AP| \cdot |AD|$. Hence, $90 - \angle AMP = 90 - \angle ADI$ or equivalently,

$$\angle ADB = \angle PMF. \tag{1}$$

Another observation is that $\angle AMQ = 90^o$. Without loss of generality, suppose Q is on ray FE. Since $\angle APQ = 90^o$, $APMQ$ is cyclic. Therefore, $\angle GAD = \angle GAP = \angle PMF = \angle ADB$ by (1). Hence,

$$AQ||BC.$$

Next, we prove that ΔARE and ΔASF are isosceles. This is true since $\angle AER = \angle CED = \angle CDE = \angle ARE$ (by $AQ||BC$). Therefore, $|AR| = |AE|$. Similarly, $|AS| = |AF|$. Since $|AE| = |AF|$, we conclude that $|AR| = |AS|$.

Source: Korean Mathematical Olympiad 2006

4. For a positive integer a , let S_a be the set of primes p for which there exists an odd integer b such that p divides $(2^{2^a})^b - 1$. Prove that for every a, there exist infinitely many primes that are not contained in S_a.

Solution: Let $F_n = 2^{2^n} + 1$. We recall some properties regarding these numbers. Note that

$$F_n = \prod_{j=1}^{n-1} F_j + 2$$

for all positive integers n. Since F_n is odd for each n, this implies that

$$gcd(F_m, F_n) = 1, (1)$$

for all distinct m, n and
$$F_m \mid F_n - 2, (2)$$
for all $m < n$. We now return to our problem.

Let a be an integer. Then $p \in S_a$ if and only if $p \mid (2^{2^a})^b - 1$ for some odd positive integer b. Suppose p is a prime number such that $p \mid F_n$ for some $n > a$. I claim that $p \in S_a$ for only finitely many such p. This finishes the problem since by (1), there are infinitely many primes p such that $p \mid F_n$ for some $n > a$.

Suppose $p \in S_a$ and $p \mid F_n$ for some $n > a$. Then $p \mid F_m - 2$ for all $m > n$. Since $p \mid (2^{2^a})^b - 1$ and $p \mid 2^{2^m} - 1$,
$$p \mid gcd(2^{2^a \cdot b} - 1, 2^{2^m} - 1) = 2^{gcd(2^a, 2^m)} - 1.$$
Since $m > n > a$, $p \mid 2^{2^a} - 1$. This latter number contains finitely many prime divisors. Therefore, there are finitely many choices for p such that $p \in S_a$ and $p \mid F_n$. This completes the problem. $\square$

Source: Korean Mathematical Olympiad 2006

5. Find the largest positive integer k such that for any ten points in the plane with the property that every subset of five points contains at least four concyclic points, there exist k concyclic points.

Solution: The answer is $k = 9$.

Consider the ten points, nine of which are the vertices of a cyclic 9-gon and the tenth point is the centre of the circle. Clearly, for any choice of five points, at least four of them are concyclic. Hence, $k \leq 9$.

Clearly $k \geq 4$. We first prove that $k \geq 5$. Let A, B, C, D be four conyclic points. Let E, F, G, H, I, J be the remaining six points. Consider five points A, B, C, P, Q where $P \neq Q$ and $P, Q \in \{E, F, G, H, I, J\}$. Four of these points are concyclic. Two of these points are P and Q. Otherwise, A, B, C and one of P, Q is conyclic, implying A, B, C, D and one of P, Q is concyclic, implying $k \geq 5$. Hence, two of these points are P and Q and the other two points are chosen from $\{A, B, C\}$. Since there are 15 choices for (P, Q), there exists two points A', B' from $\{A, B, C\}$ such that A', B', P, Q are concyclic for five distinct pairs (P, Q) from $\{E, F, G, H, I, J\}$. These five pairs consists of two pairs that overlap.. Hence, A', B' and three other points are concyclic. Therefore, $k \geq 5$.

Let A, B, C, D, E be concyclic points and X, Y are not on the circle. Consider the points A, B, C, X, Y. Four of these points are concyclic and two of them are X, Y. Without loss of generality, suppose A, B, X, Y are concyclic. Now consider A, C, D, X, Y. Four of these are concyclic and two of them are X, Y. If A is one of these points, then C or D is the fourth point, say C. Then A, B, C, X are conyclic and X lies on the circle. Impossible. Therefore, C, D, X, Y are concyclic. Finally, consider A, D, E, X, Y. Four of these points are concyclic and two of them are X, Y. If A, D, X, Y are concyclic, then A, C, D, X are concyclic and X lies on the circle. If A, E, X, Y are concyclic, then A, B, E, X, Y are concyclic and X, Y lies on the circle. A similar conclusion holds if D, E, X, Y are concyclic. Therefore, there cannot be two points outside of five concyclic points. Since five concyclic points exist, $k \leq 9$. From our construction in the first paragraph, we conclude $k = 9$.

Source: Iran Mathematical Olympiad 1997

2009 Mock Olympiad 7: (CMO Mock)
Time: 3 Hours

1. Consider the inequality

$$(a_1 + a_2 + \cdots + a_n)^2 \geq 4(a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_{n-1}a_n + a_na_1), n \geq 3.$$

Find all positive integers n for which this inequality is true for all real numbers $a_1, a_2, \cdots, a_n$.

2. Let AB be a chord, which is not a diameter, of a circle ω that has centre O. Let D be a point on ray AB past B. Let ω_1, ω_2 be circles passing through D and tangent to ω at A, B, respectively. Let P be the intersection of ω_1 and ω_2 which is not D. Prove that $\angle OPD = 90^o$.

3. The entries of a 2010×2010 board are chosen from $\{0, 1\}$ such that each row and each column contains an odd number of 1's. Suppose the board is coloured like a chessboard. Prove that the number of white squares with a 1 on it, is even.

4. Let k be a positive integer. Let $a_1, a_2, \cdots, a_k$ be positive integers and $d = gcd(a_1, a_2, \cdots, a_k)$ and $n = a_1 + a_2 + \cdots + a_k$. Prove that

$$\frac{d \cdot (n-1)!}{a_1!a_2! \cdots a_k!}$$

is an integer.

5. Let $n \geq 2$ be a positive integer and X be a set with n elements. Let $A_1, A_2, \cdots, A_{101}$ be subsets of X such that the union of any 50 of these subsets has more than $50n/51$ elements. Prove that amongst these 101 subsets, there exist three subsets such that any two of them have a common element.

2009 Mock Olympiad 7: (CMO Mock)
Time: 3 Hours

1. Consider the inequality

$$(a_1 + a_2 + \cdots + a_n)^2 \geq 4(a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_{n-1}a_n + a_na_1), n \geq 3.$$

Find all positive integers n for which this inequality is true for all real numbers $a_1, a_2, \cdots, a_n$.

Solution: The answer is $n = 4$. If $n = 3$, setting $a_1 = a_2 = a_3 = 1$ gives a contradiction since $(a_1+a_2+a_3)^2 = 9$ and $4(a_1a_2+a_2a_3+a_3a_1) = 12$. If $n \geq 5$ and n is odd, then let $n = 2k+1$. Let $a_1 = a_2 = \cdots = a_k = 1, a_{k+1} = 0, a_{k+2} = a_{k+3} = \cdots = a_{2k+1} = -1$. Then the left-hand side is zero and the right-hand side is $(k-1)+(k-1)-1 = 2k-3 > 0$. If $n \geq 5$ and n is even, let $n = 2k$. Let $a_1 = a_2 = \cdots = a_k = 1$ and $a_{k+1} = a_{k+2} = \cdots = a_{2k} = -1$. Then the left-hand side is zero and the right-hand side is $(k-1)+(-1)+(k-1)+(-1) = 2k-4 > 0$. Finally if $n = 4$, we have $(a_1+a_2+a_3+a_4)^2 \geq 4(a_1a_2+a_2a_3+a_3a_4+a_4a_1) = 4(a_1+a_3)(a_2+a_4)$. Let $x = a_1 + a_3$ and $y = a_2 + a_4$. Then the inequality is equivalent to $(x+y)^2 \geq 4xy$, which is true. $\square$

Source: Italian Mathematical Olympiad 2008

2. Let AB be a chord, which is not a diameter, of a circle ω that has centre O. Let D be a point on ray AB past B. Let ω_1, ω_2 be circles passing through D and tangent to ω at A, B, respectively. Let P be the intersection of ω_1 and ω_2 which is not D. Prove that $\angle OPD = 90^o$.

Solution: Let O_1, O_2 be the centres of ω_1, ω_2 respectively. By the tangency property, A, O, O_1 are collinear and B, O, O_2 are collinear. Therefore, since $OA = OB$ and $O_1A = O_1D$, we have that $OB||O_1D$. Furthermore, $\angle O_2DB = \angle O_2BD = \angle OBA = \angle OAB$. therefore, $AO_1||DO_2$. Hence, OO_1DO_2 is a parallelogram. Let E be the midpoint of OD, which hence is the midpoint of O_1O_2. Since $O_1D = O_1P$, we conclude that $O_1O_2 \perp PD$. Since E is on O_1O_2, we have that $EP = ED = EO$. Therefore, E is the centre of the circle with diameter OD which passes through P. We conclude that $\angle OPD = 90^o$. $\square$

Source: In Polya's Footstep by Ross Honsberger

3. The entries of a 2010×2010 board are chosen from $\{0, 1\}$ such that each row and each column contains an odd number of 1's. Suppose the board is coloured like a chessboard. Prove that the number of white squares with a 1 on it, is even.

Solution: Without loss of generality, suppose the top-left square of the board is white. Let W_1 be the white squares in the odd numbered rows and odd numbered columns, W_2 the white squares in the even numbered row and even numbered column, B_1 be the black squares in the odd numbered rows and even numbered columns and B_2 be the black squares in the even numbered rows and odd numbered columns. Note $W_1 \cup B_1$ are the odd numbered rows, of which are there are an odd number of them (1005 to be exact). Hence, the number of ones in $W_1 \cup B_1$ is odd. Similarly the number of ones in each of $W_1 \cup B_2$, $W_2 \cup B_1$, $W_2 \cup B_2$ is odd. Then the number of ones in W_1 has a different parity than the number of ones in B_1, which has a different parity than the number of ones in W_2. Hence, the number of ones in W_1 and that in W_2 have the same parity, implying the number of ones in $W_1 \cup W_2$ is even. $\square$

Source: 102 Combinatorics Problems by Titu Andreescu and Zuming Feng

4. Let k be a positive integer. Let $a_1, a_2, \cdots, a_k$ be positive integers and $d = gcd(a_1, a_2, \cdots, a_k)$ and $n = a_1 + a_2 + \cdots + a_k$. Prove that

$$\frac{d \cdot (n-1)!}{a_1! a_2! \cdots a_k!}$$

is an integer.

Solution: Let $u_1, \cdots, u_k$ be integers such that $u_1 a_1 + \cdots u_k a_k = d$. Then

$$\frac{d \cdot (n-1)!}{a_1! a_2! \cdots a_k!} = \frac{(u_1 a_1 + \cdots + u_k a_k) \cdot (n-1)!}{a_1! a_2! \cdots a_k!} = \sum_{i=1}^{k} u_i \cdot \frac{(n-1)!}{a_1! a_2! \cdots a_{i-1}! (a_i - 1)! a_{i+1}! \cdots a_k!},$$

which is an integer since $a_1 + a_2 + \cdots + a_{i-1} + (a_i - 1) + a_{i+1} + \cdots + a_k = n - 1$. $\square$

Source: Romanian Masters of Mathematics 2009

5. Let $n \geq 2$ be a positive integer and X be a set with n elements. Let $A_1, A_2, \cdots, A_{101}$ be subsets of X such that the union of any 50 of these subsets has more than $50n/51$ elements. Prove that amongst these 101 subsets, there exist three subsets such that any two of them have a common element.

Solution: We represent this by a graph. Let $A_1, A_2, \cdots, A_{101}$ be the vertices and A_i be adjacent to A_j if and only if $A_i \cap A_j$ is non-empty. It suffices to prove that this graph contains a triangle. Note that if $\deg A_i \leq 50$, then there are 50 sets not adjacent to A_i. The union of these 50 sets contains more than $50n/51$ elements. Since A_i is disjoint from these 50 sets, A_i contains at most $n/51$ elements. Hence, if $\deg A_i \leq 50$, then A_i contains at most $n/51$ elements. If there are 50 such sets, then the union of these 50 sets contains at most $50n/51$ elements, which is impossible. Therefore, $\deg A_i \leq 50$ for at most 49 sets. Therefore, $\deg A_i \geq 51$ for at least 52 sets. Since for each such A_i, there are at most 49 sets not adjacent to it, there must be two adjacent sets A_i, A_j such that $\deg A_i, \deg A_j \geq 51$. Each of A_i, A_j are each adjacent to at least 50 sets not in $\{A_i, A_j\}$. Since there are 99 sets remaining, these two sets are adjacent to a common set. This forms a triangle in the graph, as desired. $\square$

Source: Romanian Team Selection Test 2004

2009 Mock Olympiad 8: (CMO Mock)
Time: 3 Hours

1. Let two circles C_1, C_2 with different radii be externally tangent at a point T. Let A be on C_1 and B be on C_2, with $A, B \neq T$ such that $\angle ATB = 90^o$.

   (a) Prove that all such lines AB are concurrent.

   (b) Find the locus of the midpoints of all such segments AB.

2. Let S be a set with 2009 elements. Find the smallest positive integer k such that for any subsets $A_1, A_2, \cdots, A_k$ of S, for each $i \in \{1, 2, \cdots, k\}$ we can choose either $B_i = A_i$ or $B_i = S - A_i$ such that
$$\bigcup_{i=1}^{k} B_i = S.$$

3. Let ABC be a triangle with an interior point P such that $\angle BPC = 90^o$ and $\angle BAP = \angle BCP$. Let M, N be midpoints of AC, BC, respectively. Suppose $BP = 2PM$. Prove that A, P, N are collinear.

4. Find all pairs of positive integers (n, k) such that the following statement is true: there exist integers a, b such that $gcd(a, n) = 1, gcd(b, n) = 1$ and $a + b = k$.

5. Let a, b, c be fixed positive real numbers. Find all triples of positive real numbers (x, y, z) such that $x + y + z = a + b + c$ and
$$4xyz = a^2x + b^2y + c^2z + abc.$$

2009 Mock Olympiad 8: (CMO Mock)
Time: 3 Hours

1. Let two circles C_1, C_2 with different radii be externally tangent at a point T. Let A be on C_1 and B be on C_2, with $A, B \neq T$ such that $\angle ATB = 90^o$.

   (a) Prove that all such lines AB are concurrent.

   (b) Find the locus of the midpoints of all such segments AB.

   Solution: (a) Let O_1, O_2 be the centres of C_1, C_2, with radii r_1, r_2 respectively, with $r_1 < r_2$. Since $\angle ATB = 90^o$, we have $\angle O_1TA + \angle O_2TB = 90^o$. Therefore, $\angle TO_1A + \angle TO_2B = 180 - 2\angle O_1TA + 180 - 2\angle O_2TB = 180^o$. Therefore, $O_1A||O_2B$. Let AB intersect O_1O_2 at X. Then $XO_1/XO_2 = O_1A/O_2B = r_1/r_2$. Since X is on ray O_2O_1 beyond O_1, the point X is independent of A, B. Hence, X is the desired concurrent point.

   (b) Let M be the midpoint of AB. Let N be the midpoint of O_1O_2. Then $MN = \frac{1}{2}(r_1+r_2) = NO_1 = NO_2$. Therefore, M lies on the circle with centre N passing through O_1, O_2. Since $A, B \neq T$ and A, B are on the same side of the line O_1O_2, M cannot be O_1 or O_2. Conversely, let M be any point on the circle with centre N passing through O_1, O_2, which is not O_1 or O_2. Let A, B be points on the same side of O_1O_2 as M such that $O_1A||NM||O_2B$. Then M is the midpoints of AB. Since $O_1A||O_2B$, we have that $\angle AO_1T + \angle TO_2B = 180^o$, which implies $\angle O_1TA + \angle O_2TB = 90 - (\angle AO_1T/2) + 90 - (\angle TO_2B/2) = 90$. This implies $\angle ATB = 90^o$. Hence, all M is the midpoint of AB where $\angle ATB = 90^o$. Therefore, the locus of such midpoints is the circle with diameter O_1O_2 excluding the points O_1 and O_2. $\square$

   Source: Hong Kong Team Selection Test 2008

2. Let S be a set with 2009 elements. Find the smallest positive integer k such that for any subsets $A_1, A_2, \cdots, A_k$ of S, for each $i \in \{1, 2, \cdots, k\}$ we can choose either $B_i = A_i$ or $B_i = S - A_i$ such that

$$\bigcup_{i=1}^{k} B_i = S.$$

   Solution: Since S has 2009 elements, at least one of $A_1, S - A_1$ contains at least half of S. Choose B_1 such that $|B_1| \geq |S|/2$. Iteratively define $S_i = S - B_1 - B_2 \cdots - B_i$. Choose B_{i+1} such that B_{i+1} contains at least half of the elements from S_i. Since $|S_{i+1}| \leq |S_i| < 2$, eventually S_k is empty. This happens when $k = \lceil \log_2(2009 + 1) \rceil = 11$. Therefore, $k \leq 11$.

   Suppose $k \leq 10$. Then we choose $A_1, A_2, \cdots, A_k$ such that there exist at least one element in each of the 2^k regions of the Venn diagram for $A_1, \cdots, A_k$. This is possible since $2009 > 2^{10}$. Then after choosing $B_1, B_2, \cdots, B_k$, there is one region in the Venn diagram missing. The element in this region is not in $B_1 \cup B_2 \cup \cdots \cup B_k$, so this union cannot equal S. Therefore, $k = 11$.

   Source: Brazilian Mathematical Olympiad 2003

3. Let ABC be a triangle with an interior point P such that $\angle BPC = 90^o$ and $\angle BAP = \angle BCP$. Let M, N be midpoints of AC, BC, respectively. Suppose $BP = 2PM$. Prove that A, P, N are collinear.

   Solution: Let X be the midpoint of PC and Y be the midpoints of PB. Then $MX||AP$, $NX||PB$. Furthermore, $|MX| = \frac{1}{2}|AP|$ and $|XN| = \frac{1}{2}|PB|$. Therefore, $\angle BAP = \angle NMX$. But since $\angle BPC = 90^o$, we have that $\angle NPC = \angle NCP = \angle BCP = \angle BAP$. Therefore, $PMXN$ are cyclic. Note that $PXNY$ is a rectangle. Therefore, $PMXNY$ is cyclic.

   Since $|BP| = 2|PM|$, we have $|PM| = |PY| = |XN|$. Since $PMXNY$ is cyclic, we have that $\angle PXM = \angle NPX$. This implies $MX||PN$. But since $MX||AP$, we conclude that A, P, N are collinear. $\square$

   Source: Indian Mathematical Olympiad 2009

4. Find all pairs of positive integers (n, k) such that the following statement is true: there exist integers a, b such that $gcd(a, n) = 1, gcd(b, n) = 1$ and $a + b = k$.

   Solution: The answer is all (n, k) such that n is odd, or n is even and k is even.

   If n is even and k is odd, then since $gcd(a, n) = gcd(b, n) = 1$, we have that a, b are odd. Therefore, $a + b$ is even. This implies k cannot be odd.

   If n is odd, let $p_1, \cdots, p_t$ be the prime divisors of n. Note that $p_i \geq 3$ for each i. Let a be chosen such that $a \not\equiv 0, k \bmod p_i$. This is possible by Chinese Remainder Theorem and $p_i \geq 3$. Therefore, $gcd(a, n) = 1$ and $gcd(k - a, n) = 1$. Therefore, setting $b = k - a$ yields $a + b = k$.

   If n is even and k is even, let $2, p_1, \cdots, p_t$ be the prime divisors of n. Then note $k \equiv 0 \bmod 2$. Then similarly, we can choose a such that $a \not\equiv 0, k \bmod p_i$. Therefore, $gcd(a, n) = 1$ and $gcd(k - a, n) = 1$. Setting $b = k - a$ yields $a + b = k$. This finishes the problem. $\square$

   Source: IberoAmerican Mathematical Olympiad 2004

5. Let a, b, c be fixed positive real numbers. Find all triples of positive real numbers (x, y, z) such that $x + y + z = a + b + c$ and

$$4xyz = a^2x + b^2y + c^2z + abc.$$

   Solution: We recall that $u^2+v^2+w^2+2uvw = 1$ if and only if $u = \cos A, v = \cos B, w = \cos C$ where A, B, C are angles of a triangle. Note that

$$\frac{a^2}{4yz} + \frac{b^2}{4zx} + \frac{c^2}{4xy} + \frac{abc}{4xyz} = 1.$$

Therefore, there exist angles of a triangle A, B, C such that

$$\cos A = \frac{a}{2\sqrt{yz}}, \cos B = \frac{b}{2\sqrt{zx}}, \cos C = \frac{c}{2\sqrt{xy}}. (*)$$

Therefore,

$$x + y + z = a + b + c = 2(\sqrt{yz}\cos A + \sqrt{zx}\cos B + \sqrt{xy}\cos C).$$

I now claim that

$$x + y + z \geq 2(\sqrt{yz}\cos A + \sqrt{zx}\cos B + \sqrt{xy}\cos C).$$

Let $C = \pi - A - B$. Then this inequality becomes equivalent to

$$(\sqrt{z} - (\sqrt{x}\cos B + \sqrt{y}\cos A))^2 + (\sqrt{x}\sin B - \sqrt{y}\sin A)^2 \geq 0$$

and equality holds if and only if

$$\sqrt{z} = \sqrt{x}\cos B + \sqrt{y}\cos A \text{ and } 0 = \sqrt{x}\sin B - \sqrt{y}\sin A.$$

Squaring these two equations and adding them and using the relation in (*) yields $c = x+y-z$. Similarly, $a = y + z - x$ and $b = z + x - y$. Solving this for x, y, z yields

$$x = \frac{b+c}{2}, y = \frac{c+a}{2}, z = \frac{a+b}{2}.$$

This is easily verified as a solution to the equation since clearly $x + y + z = a + b + c$ and

$$4xyz - a^2x - b^2y - c^2z - abc = 4 \cdot \frac{b+c}{2} \cdot \frac{c+a}{2} \cdot \frac{a+b}{2} - a^2 \cdot \frac{b+c}{2} - b^2 \cdot \frac{c+a}{2} - c^2 \cdot \frac{a+b}{2} - abc = 0.$$

Source: IMO Shortlist 1995

2009 Mock Olympiad 9: (USAMO Mock)
Time: 4.5 Hours

1. A positive integer m is said to be a left-block of a positive integer n if m occurs as the left-most digits of n in decimal representation. For example, 137 is a left-block of 13729. But 1373 is not a left-block of 13729. An integer is a left-block of itself. Let $a_1, a_2, \cdots a_t$ be a finite sequence of positive integers such that no number in the sequence is the left-block of another number in the sequence. Find the maximum possible value of
$$\sum_{k=1}^{t} \frac{1}{a_k}.$$

2. Let x, y, z be positive real numbers in the range $[1, 2]$. Prove that
$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 6\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right).$$
Determine when equality holds.

3. Given an acute-angled triangle ABC, a line l intersect the lines BC, CA, AB at D, E, F respectively, where E, F lie on their respective sides. Let O_1, O_2, O_3 be the circumcentres of $\Delta AEF, \Delta BFD, \Delta CDE$, respectively. Prove that the orthocentre of $\Delta O_1O_2O_3$ lies on l.

2009 Mock Olympiad 9: (USAMO Mock)
Time: 4.5 Hours

1. A positive integer m is said to be a left-block of a positive integer n if m occurs as the left-most digits of n in decimal representation. For example, 137 is a left-block of 13729. But 1373 is not a left-block of 13729. An integer is a left-block of itself. Let $a_1, a_2, \cdots a_t$ be a finite sequence of positive integers such that no number in the sequence is the left-block of another number in the sequence. Find the maximum possible value of

$$\sum_{k=1}^{t} \frac{1}{a_k}.$$

Solution: The answer is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$ and is attained when $t = 9$ and $a_k = k$ for $k = 1, 2, \cdots, 9$.

We will prove this by induction on t. If $t \leq 9$, then the maximum of the sum is clear since no two numbers in the sequence are the same. The maximum holds when $t = 9$ and $a_k = k$ for $k = 1, 2, \cdots, 9$. Now suppose $t > 9$. Let $b_1, \cdots, b_{10}$ be the 10 largest numbers in the sequence with $b_1 < b_2 < \cdots < b_{10}$.

If $b_1 \geq 10$, then $\lfloor b_1/10 \rfloor$ is not in the sequence since $\lfloor b_1/10 \rfloor$ is the left-block of b_1. Then

$$\frac{1}{\lfloor b_1/10 \rfloor} > \frac{1}{b_1} + \frac{1}{b_1+1} + \cdots + \frac{1}{b_1+9} \geq \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{10}}.$$

Hence the sequence with $b_1, \cdots, b_{10}$ replaced by $\lfloor b_1/10 \rfloor$ is a sequence satisfying the desired property with $t - 9$ elements. By induction hypothesis, this sum is at most $1 + \frac{1}{2} + \cdots + \frac{1}{9}$.

If $b_1 < 10$, then let $c_1 < \cdots < c_m$ be the numbers in the sequence greater than or equal 10. Note $1 \leq m \leq 9$ since there are at least 10 numbers in the sequence. Then since $\lfloor c/10 \rfloor$ is not in the sequence, we have that

$$\frac{1}{\lfloor c_1/10 \rfloor} > \frac{1}{c_1} + \cdots + \frac{1}{c_m}.$$

Then replacing $c_1, \cdots, c_m$ with $\lfloor c_1/10 \rfloor$ leaves a sequence with $t - m$ terms with the desired property. By induction hypothesis, the sum is at most $1 + \frac{1}{2} + \cdots + \frac{1}{9}$. $\square$

Source: Iran Mathematical Olympiad 1998

2. Let x, y, z be positive real numbers in the range $[1,2]$. Prove that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 6\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right).$$

Determine when equality holds.

Solution: Let

$$f(x,y,z) = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) - 6\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right).$$

Without loss of generality, suppose $x \geq y \geq z$. Let $t = (y+z)/2$. Note that $x \geq t$. We will prove that $f(x,y,z) \geq f(x,t,t) \geq 0$.

Then $f(x,y,z) - f(x,t,t)$

$$\begin{aligned}
&= (x+y+z)(\tfrac{1}{x}+\tfrac{1}{y}+\tfrac{1}{z}) - (x+t+t)(\tfrac{1}{x}+\tfrac{1}{t}+\tfrac{1}{t}) - 6(\tfrac{x}{y+z}+\tfrac{y}{z+x}+\tfrac{z}{x+y}-\tfrac{x}{2t}-\tfrac{t}{t+x}-\tfrac{t}{x+t}) \\
&= (x+y+z)(\tfrac{1}{y}+\tfrac{1}{z}-\tfrac{2}{t}) - 6(\tfrac{y}{z+x}+\tfrac{z}{x+y}-\tfrac{2t}{x+t}) \\
&= \text{lots of work that you will do} \\
&= (y-z)^2(x+y+z)(\tfrac{1}{yz(y+z)} - \tfrac{3}{(x+y)(x+z)(x+t)}).
\end{aligned}$$

Using the fact that $x \geq \frac{y+z}{2}$, we get that this expression is greater than or equal to

$$\frac{1}{4}(y-z)^2(x+y+z)\left(\frac{3y^3+y^2z+yz^2+3z^3}{yz(y+z)(x+y)(x+z)(x+t)}\right) \geq 0$$

with equality if and only if $y = z$. Therefore, $f(x,y,z) \geq f(x,t,t)$ with equality if and only if $y = z$.

Now we prove that $f(x,t,t) \geq 0$. We have that

$$f(x,t,t) = \text{ lots of work } = \frac{-(x-t)^2(x-2t)}{xt(x+t)}.$$

Since $x, y, z, t \in [1,2]$, we have that $x - 2t \leq 0$. Therefore, $f(x,y,z) \geq f(x,t,t)$ with equality if and only if $x = 2, t = 1$ or $x = t$. Therefore, $f(x,y,z) \geq f(x,t,t) \geq 0$ with equality if and only if $x = y = z$ or $(x,y,z) = (2,1,1), (1,2,1), (1,1,2)$. $\square$

Source: Vietnamese Mathematical Olympiad 2003

3. Given an acute-angled triangle ABC, a line l intersect the lines BC, CA, AB at D, E, F respectively, where E, F lie on their respective sides. Let O_1, O_2, O_3 be the circumcentres of $\Delta AEF, \Delta BFD, \Delta CDE$, respectively. Prove that the orthocentre of $\Delta O_1O_2O_3$ lies on l.

Solution: I claim that the circumcircles of $\Delta AEF, BFD, CDE, ABC$ are concurrent at a point P. (This is actually a version of Miquel's Theorem. But the proof is presented here.)

In the first case, without loss of generality suppose E, F are the points on the sides CA, AB, respectively and F be on ray CB. Let P be the intersection of the circumcircles of ΔAEF and ΔBFD. Then

$$\angle BPD = \angle BFD = \angle AFE = \angle APE.$$

Furthermore,

$$\angle FPD = \angle ABC \text{ and } \angle EPF = \angle EAF = \angle BAC.$$

Therefore, $\angle EPD = \angle EPF + \angle FPD = \angle BAC + \angle ABC$. This implies $\angle EPD + \angle ECD = \angle EPD + \angle ACB = 180^o$. Therefore, $PECD$ is cyclic. This implies P lies on the circumcircle of ΔCDE. Furthermore, this also implies $\angle CDE = \angle CPE$. Therefore,

$$\angle APC = \angle APE + \angle EPC = \angle BFD + \angle FDB = \angle ABC.$$

this implies $APBC$ is cyclic. Therefore, P lies on the circumcircle of ΔABC.

Therefore, O_1, O_2, O_3 are the circumcentres of $\Delta PEF, PFD, PDE$, respectively. Note that D, F, E are collinear, in that order. We now prove that the orthocentre of $\Delta O_1O_2O_3$ lies on this point. Let X, Y, Z be the feet of the altitude from O_1, O_2, O_3, respectively in triangle $\Delta O_1O_2O_3$. Since O_1X and PD are both perpendicular to O_2O_3, we have that $O_1X||PD$. similarly, $O_2Y||PE$ and $O_3Z||PF$. Let O_3Z intersects l at H. It suffices to prove that O_1, H, X are collinear and O_2, H, Y are collinear. Let $\angle DPE = \alpha, \angle PDE = \beta, \angle PED = \gamma$. Let $\angle DPF = \theta$. Hence, $\angle EPF = \alpha - \theta$. Let U, V be the midpoints of PD, PF, respectively. Then since PUO_2V is cyclic, we have that $\theta = \angle DPF = \angle O_1O_2O_3$, which implies $\angle ZHX = 180 - \theta$. However, since $O_1X||PD$ and $O_3Z||PF$, it implies that O_3Z intersect O_1X at an angle of θ. But $\angle ZHX = 180 - \theta$. This implies O_1, H, X are collinear. Similarly, O_2, H, Y are collinear. Therefore, the orthocentre of $\Delta O_1O_2O_3$ lies on l, as desired. $\square$

Source: China Team Selection Test 2008

2009 Mock Olympiad 10: (USAMO Mock)
Time: 4.5 Hours

1. Given a triangle ABC and an interior point P, prove that

$$\min\{|PA|, |PB|, |PC|\} + |PA| + |PB| + |PC| < |AB| + |BC| + |CA|.$$

2. At a math contest, there are $2n$ students participating. Each of them submits a problem to the jury, which thereafter gives each student one of the $2n$ problems submitted. We say that the contest is fair if there are n participants who receive their problems from the other n participants. Prove that the number of distributions of the problems that results in a fair contest is a perfect square. [2]

3. Let x, y, z be real numbers such that $x + y + z = xy + yz + zx$. Find the minimum possible value of

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1}.$$

[2]Just to be clear, each student receives a unique problem.

2009 Mock Olympiad 10: (USAMO Mock)
Time: 4.5 Hours

1. Given a triangle ABC and an interior point P, prove that

$$\min\{|PA|, |PB|, |PC|\} + |PA| + |PB| + |PC| < |AB| + |BC| + |CA|.$$

Solution: Let D, E, F be midpoints of BC, CA, AB respectively. Note that P is in at least two of the quadrilaterals $ABDE, BCEF, CAFD$. Without loss of generality, suppose that P is in $ABDE$ and $BCEF$. I claim that $|AP| + |PB| < |AE| + |ED| + |DB|$. We know that AP intersects either side BD or D. If AP intersect side BD, say at Q, then $|PA| + |PB| < |PA|+|PQ|+|QB| = |AQ|+|QB| < |AE|+|ED|+|DQ|+|QB| = |AE|+|ED|+|DB|$. If AP intersects DE, say at R, then $|PA|+|PB| < |PA|+|PR|+|RD|+|DB| = |AR|+|RD|+|DB| < |AE| + |ER| + |RD| + |DB| = |AE| + |ED| + |DB|$, as desired.

Similarly, $|BP| + |PC| < |BF| + |FE| + |EC|$. Adding these two equations yield

$$|BP|+|PA|+|PB|+|PC| < |AE|+|EC|+|EF|+|BD|+|ED|+|BF| = |AB|+|BC|+|CA|.$$

Therefore,

$$\min\{|PA|, |PB|, |PC|\} + |PA| + |PB| + |PC| < |AB| + |BC| + |CA|.$$

as desired. □

Source: IMO Shortlist 1999

2. At a math contest, there are $2n$ students participating. Each of them submits a problem to the jury, which thereafter gives each student one of the $2n$ problems submitted. We say that the contest is fair if there are n participants who receive their problems from the other n participants. Prove that the number of distributions of the problems that results in a fair contest is a perfect square. [3]

Solution: We label the people $1, 2, \cdots, 2n$. We construct a directed graph where each person is represented by a vertex and an edge is drawn from person i to person j if and only if the problem proposed by person i is sent to person j or vice versa. There is a 1-1 correspondence between the distributions of a fair contest and such a graph with a perfect matching. But such a graph has $2n$ vertices and $2n$ edges and every vertex has degree two. Hence, such a graph is a collection of directed cycles. Furthermore, such a graph has a perfect matching if and only if each cycle has even length. Hence, it suffices to find the number of graphs, with labelled vertices consisting of even vertices.

Let S be the set of pairings of $\{1, 2, \cdots, 2n\}$ into n pairwise disjoint pairs. We will construct a $1 - 1$ correspondence between S^2 and labelled directed graphs consisting entirely of even

[3] Just to be clear, each student receives a unique problem.

directed cycles. Let $(M_1, M_2) \in S^2$. Let ψ map (M_1, M_2) to the graph corresponding formed by the union of M_1 and M_2 where the edges from M_1 are coloured red and the edges from M_2 are coloured blue. For each component (which is an even cycle), we choose the smallest number and put an arrow leaving this smallest number along the red edge. This determines the direction of all cycles in the graph. I claim that ψ gives a one-to-one correspondence between S^2 and labelled directed graphs consisting of even cycles. Given such a graph, for each component, consider the smallest number and the edge that leaves this number. We colour this edge red. The remaining edges can then be uniquely coloured red and blue so that no two edges of the same colour are adjacent. This gives us a matching M_1 of red edges and M_2 of blue edges. Clearly, $\psi(M_1, M_2)$ gives us the original graph. Therefore, the number of such graphs is $|S|^2$ which is a perfect square. (This number is in fact $1^2 \cdot 3^2 \cdots (2n-1)^2$.) $\square$

Source: Romanian Team Selection Test 2003

3. Let x, y, z be real numbers such that $x + y + z = xy + yz + zx$. Find the minimum possible value of
$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1}.$$

Solution: The answer is $-1/2$ and occurs at $x = 1, y = -1, z = -1$ and its permutations.

Let $a = x + y + z = xy + yz + zx$ and $b = xyz$. Then the inequality
$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} \geq -\frac{1}{2}$$
is equivalent to
$$2\sum_{cyc} x(y^2+1)(z^2+1) \geq -(x^2+1)(y^2+1)(z^2+1)$$
$$2xyz(xy+yz+zx) + 2(\sum_{sym} x^2y) + 2(x+y+z) \geq -x^2y^2z^2 - \sum_{cyc} x^2y^2 - \sum_{cyc} x^2 - 1$$
$$2xyz(xy+yz+zx) + 2(x+y+z)(xy+yz+zx) - 6xyz + 2(x+y+z)$$
$$\geq -x^2y^2z^2 - (xy+yz+zx)^2 + 2xyz(x+y+z) - (x+y+z)^2 + 2(xy+yz+zx) - 1.$$
Hence.
$$b^2 + 4a^2 - 6b \geq 1.$$
We can write this as
$$x^2y^2z^2 - 6xyz + 2(x+y+z)^2 + 2(xy+yz+zx)^2 + 1 \geq 0.$$
By a lot of work and creativity, we can write this as
$$(x-yz)^2 + (y-zx)^2 + (z-xy)^2 + (1+xy+yz+zx)^2 + (x+y+z+xyz)^2 \geq 0$$
which is clearly true. $\square$

Source: Brazilian Mathematical Olympiad 2005

2009 Mock Olympiad 11: (USAMO Mock)
Time: 4.5 Hours

1. Let S be a set of 2009 elements and $P(S)$ be the set of all subsets of S. Let $f : P(S) \to \mathbb{R}$ be a function such that
$$f(X \cap Y) = \min\{f(X), f(Y)\}$$
for all $X, Y \in P(S)$. Find the maximum possible number of elements in the range of f.

2. Let $\mathcal{P}$ be a convex polygon. Prove that $\mathcal{P}$ contains an interior point Q such that for every line l passing through Q, if we let X, Y be the two intersection points of l and the boundary of $\mathcal{P}$, we have
$$\frac{1}{2} \leq \frac{|QX|}{|QY|} \leq 2.$$

3. Let m, n be positive integers. Prove that there exists a positive integer k such that $2^k - m$ contains at least n distinct prime divisors.

2009 Mock Olympiad 11: (USAMO Mock)
Time: 4.5 Hours

1. Let S be a set of 2009 elements and $P(S)$ be the set of all subsets of S. Let $f : P(S) \to \mathbb{R}$ be a function such that
$$f(X \cap Y) = \min\{f(X), f(Y)\}$$
for all $X, Y \in P(S)$. Find the maximum possible number of elements in the range of f.

Solution: The answer is 2010.

Let $S = \{1, 2, \cdots, 2009\}$. For each $i \in S$, let $S_i = S - \{i\}$. Note that for each $X \in P(S)$ with $X \neq S$, X can be written uniquely as the intersection of sets chosen from S_i. Specifically,
$$X = \bigcap_{i \notin X} S_i.$$
Note that inductively, we can prove that for any subset $\mathcal{I}$ of $\{1, 2, \cdots, 2009\}$
$$f(\bigcap_{j \in \mathcal{I}} S_j) = \min_{j \in \mathcal{I}}\{f(S_j)\}.$$
Therefore, if we are given values for $f(S_1), \cdots, f(S_{2009})$ and $f(S)$, then f is determined for all of $P(S)$. Specifically, for each $X \in P(S) - S$, $f(X)$ must take on one of the values of $f(S_1), \cdots, f(S_{2009})$. Hence by including the value for $f(S)$, the number of elements in the range of f is at most 2010. We now construct a function f that satisfies the given conditions and have 2010 elements in its range. We let $f(S) = 2010$ and for all $X \in P(S) - S$, $f(X)$ to be equal to the smallest element in $\{1, 2, \cdots, 2009\}$ not in X. The condition $f(X \cap S) = \min\{f(X), f(S)\}$ is clearly satisfied since $f(S) > f(X)$ for all $X \in P(S), X \neq S$. Finally, $f(X \cap Y)$ is the smallest element not in $X \cap Y$, which is the smallest element not in X or not in Y, which is $\min\{f(X), f(Y)\}$. The range of f is $\{1, 2, \cdots, 2010\}$. $\square$

Source: Romanian Team Selection Test 2007

2. Let $\mathcal{P}$ be a convex polygon. Prove that $\mathcal{P}$ contains an interior point Q such that for every line l passing through Q, if we let X, Y be the two intersection points of l and the boundary of $\mathcal{P}$, we have
$$\frac{1}{2} \leq \frac{|QX|}{|QY|} \leq 2.$$

Solution: For each point A on the boundary of $\mathcal{P}$, Let S_A be the image of the dilation of $\mathcal{P}$ about A by ratio $2/3$. We will prove that there exists a point Q in the interior of P such that Q is in S_A for all A on the boundary of $\mathcal{P}$. Let A, B, C be three distinct points on the boundary of $\mathcal{P}$. Let G be the centroid of ΔABC, D, E, F be the midpoints of BC, CA, AB, respectively, and A', B', C' be the intersection of AG, BG, CG with the boundary of $\mathcal{P}$. Therefore,
$$\frac{AG}{AA'} \leq \frac{AG}{AD} = 2/3.$$

Hence, $G \in S_A$. Similarly, $G \in S_B$ and $G \in S_C$. Therefore, $G \in S_A \cap S_B \cap S_C$. By Helly's Theorem, there exists a point Q is in all S_A for all A on the boundary of $\mathcal{P}$. Let l be a line passing through Q and intersecting the boundary at X, Y. Then

$$\frac{XQ}{XY} \leq \frac{2}{3} \Rightarrow \frac{|QX|}{|QX| + |QY|} \leq \frac{2}{3} \Rightarrow \frac{|QX|}{|QY|} \leq 2.$$

Similarly,

$$\frac{YQ}{YX} \leq \frac{2}{3} \Rightarrow \frac{|QX|}{|QY|} \geq \frac{1}{2}.$$

This solves the problem. $\square$

Source: Iran Mathematical Olympiad 2004

3. Let m, n be positive integers. Prove that there exists a positive integer k such that $2^k - m$ contains at least n distinct prime divisors.

Solution: By repeated division by 2, we may assume that m is odd. Suppose there exists a positive integer N such that all numbers of the form $2^k - m$ has at most N prime divisors $p_1, p_2, \cdots, p_N$. Let M be a positive integer such that $p_i^M \nmid 2^k - m$ for all $i \in \{1, 2, \cdots, N\}$. Let b be a positive integer such that $p_i^M | 2^b - 1$ for all $i \in \{1, 2, \cdots, M\}$. Then $2^{k+lb} - m$ must be of the form $p_1^{e_1} \cdots p_N^{e_N}$, by the maximality of N and since p_i must divide $2^{k+lb} - m$. We can choose l large enough so that some $e_i > M$. Then $p_i^M | 2^{k+lb} - m$. But since $p_i^M | 2^b - 1$, we conclude that $p_i^M | 2^k - m$, which contradicts the choice of M. Therefore, there is no maximum value for N. This problem is solved. $\square$

Source: China Team Selection Test 2006

2009 Mock Olympiad 12: (USAMO Mock)
Time: 4.5 Hours

1. Let $\mathbb{R}^+$ denote the positive real numbers. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a function such that $x \le y$ implies $f(x) \le f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists $t \in \mathbb{R}$ such that

$$f\left(t + \frac{1}{f(t)}\right) < 2f(t).$$

2. Prove that for all positive integers m, n with m odd, the following number is an integer:

$$\frac{1}{3^m n} \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

3. A word is a sequence of 2009 letters from the alphabet a, b, c, d. A word is said to be complicated if it contains two consecutive groups of identical letters. The words caab, baba and cababdc, for example, are complicated words, while bacba and dcbdc are not. A word that is not complicated is a simple word. Prove that the numbers of simple words with 2009 letters is greater than 2^{2009}.

2009 Mock Olympiad 12: (USAMO Mock)
Time: 4.5 Hours

1. Let $\mathbb{R}^+$ denote the positive real numbers. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a function such that $x \le y$ implies $f(x) \le f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists $t \in \mathbb{R}$ such that

$$f\left(t + \frac{1}{f(t)}\right) < 2f(t).$$

Solution: Suppose $f(x + \frac{1}{f(x)}) \ge 2f(x)$ for all $x \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$ and define

$$x_n = x_{n-1} + \frac{1}{f(x_{n-1})}$$

for all $n \in \mathbb{N}$. Then

$$f(x_n) = f\left(x_{n-1} + \frac{1}{f(x_{n-1})}\right) \ge 2f(x_{n-1}), \forall n \in \mathbb{N}.$$

Therefore, we conclude that

$$f(x_n) \ge 2^n f(x_0), \forall n \in \mathbb{N}.$$

By summing over all $k \in \{1, 2, \cdots, n\}$, we have that

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \left(x_{k-1} + \frac{1}{f(x_{k-1})}\right).$$

This implies

$$x_n = x_0 + \sum_{k=0}^{n-1} \frac{1}{f(x_k)} \le x_0 + \sum_{k=0}^{n-1} \frac{1}{2^k \cdot f(x_0)} < x_0 + \frac{2}{f(x_0)}$$

which is a constant. Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is non-decreasing and bounded above by a constant, which implies it has a least upper bound, say y. Therefore, $f(x_n) \le f(y)$ for all $n \in \mathbb{N}$. But since $f(x_n) \ge 2^n f(x_0)$, we conclude that $f(y) \ge 2^n f(x_0)$ for all $n \in \mathbb{N}$, which is impossible. Therefore, $f(t + \frac{1}{f(t)}) \le 2f(t)$ for some $t \in \mathbb{R}$.

Source: Iran Mathematical Olympiad 2002

2. Prove that for all positive integers m, n with m odd, the following number is an integer:

$$\frac{1}{3^m n} \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

Solution: Let ω be a primitive cube root of unity. Then

$$\sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k = \frac{1}{3} \cdot \left((1 + \sqrt[3]{3n-1})^{3m} + (1 + \sqrt[3]{3n-1}\omega)^{3m} + (1 + \sqrt[3]{3n-1}\omega^2)^{3m}\right).$$

Let $\alpha = 1 + \sqrt[3]{3n-1}, \beta = 1 + \sqrt[3]{3n-1}\omega, \gamma = 1 + \sqrt[3]{3n-1}\omega^2$. Then it is easy to calculate that $\alpha + \beta + \gamma = 3, \alpha\beta + \beta\gamma + \gamma\alpha = 3, \alpha\beta\gamma = 3n$. (Simply use the facts that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.) Therefore, if we let $S_k = \frac{1}{3} \cdot (\alpha^k + \beta^k + \gamma^k)$, then $S_0 = 1, S_1 = 1, S_2 = 1$ and

$$S_{k+3} = 3S_{k+2} - 3S_{k+1} + 3n \cdot S_k.$$

We need to prove that $3^m n$ divides S_{3m} for all positive integers m, n and m odd. It suffices to prove the following two facts;

(a) If we view each $S_k, k \in \mathbb{N}$ as a polynomial in n, then the constant term in S_{3m} is zero for each odd $m \in \mathbb{N}$.

(b) For each odd m, the coefficients of $S_{3m}, S_{3m+1}, S_{3m+2}, S_{3m+3}, S_{3m+4}, S_{3m+5}$ (when viewed as a polynomial in n) are divisible by 3^m.

After we prove these two facts, we can conclude that $3^m \cdot n$ divides S_{3m} for all positive integers m, n with m odd and we have solved the problem.

We will prove (a) by induction on m. Let c_i be the constant term of S_i. Then $c_{k+3} = 3c_{k+2} - 3c_{k+1}$. Since $c_0 = c_1 = c_2 = 1$, we have that $c_3 = 0$. We prove that $c_{3m} = 0$ and $c_{3m-1} = c_{3m-2}$ by induction. We have proved the case for $m = 1$. Now suppose we have proved this for a given m. Let $t = c_{3m-1} = c_{3m-2}$. Then $c_{3m+1} = -3t, c_{3m+2} = -9t, c_{3m+3} = -18t, c_{3m+4} = -27t, c_{3m+5} = -27t, c_{3m+6} = 0$. This proves (a) by induction.

We will prove (b) by induction on m. Clearly the statement is true for $m = 1$ since S_{k+3} in the recurrence relation is a multiple of 3. Suppose 3^m divides all of the coefficients of $S_{3m}, \cdots, S_{3m+5}$. Then S_{3m+6} is 3 times a linear combination of $S_{3m+3}, S_{3m+4}, S_{3m+5}$, implying 3^{m+1} divides the coefficients of S_{3m+6}. Similarly, we can prove that 3^{m+1} divides the coefficients of $S_{3m+6}, \cdots, S_{3m+11}$. This proves (b) by induction.

Source: Romanian Team Selection Test 2004

3. A word is a sequence of 2009 letters from the alphabet a, b, c, d. A word is said to be complicated if it contains two consecutive groups of identical letters. The words caab, baba and cababdc, for example, are complicated words, while bacba and dcbdc are not. A word that is not complicated is a simple word. Prove that the numbers of simple words with 2009 letters is greater than 2^{2009}.

Solution: We will prove this by induction on n. Let $f(n)$ be the number of simple words with n letters. It is easy to see that $f(1) = 4$ and $f(4) = 12$. It suffices to prove that $f(n+1) > 2f(n)$ for all $n \in \mathbb{N}$. Given the set of simple words with n letters, we can form words by appending each word in the set by one of four letters in the alphabet to form $4f(n)$ words. Let T_n be this set of $4f(n)$ words of $n+1$ letters, whose first n letters form a simple word. If a word in this set is complicated, then since the first n letters form a simple word, the last letter in the word makes the whole word complicated. There exists a minimal positive integer k such that the last two consecutive blocks of k letters are identical. Therefore, we can partition the complicated words in T_n into pairwise disjoint sets $U_1, U_2, \cdots, U_k$ where $k = \lfloor n/2 \rfloor$ where the last two consecutive blocks of k letters are identical. Each of these blocks are also simple, for otherwise the first n letters would form a complicated word, which is impossible. Therefore, $|U_i| \leq f(n-i)$, since a word in U_i can be formed by taking a simple word of length $n-i$ and adding the i letters which is the last i letters of this simple word. Therefore, the number of simple words of length $n+1$ is

$$4f(n) - \sum_{i=1}^{k} |T_i| \geq 4f(n) - \sum_{i=1}^{k} f(n-i) > 2f(n) + \sum_{i=1}^{k} (\frac{1}{2^{i-1}} f(n) - f(n-i))$$

By strong induction, since $f(n) > 2^i f(n-i)$, we conclude that this number is strictly greater than $2f(n)$, which completes the proof. $\square$

Source: Romanian Team Selection Test 2003

2009 Mock Olympiad 13: (IMO Mock)
Time: 4.5 Hours

1. Find all positive integers $n \geq 3$ such that there exists a convex polygon with n vertices on the Cartesian coordinate plane such that the side lengths are all odd integers and every vertex has integer coordinates. [4]

2. Let d, n be positive integers such that d divides n. Let S be the set of all (ordered) n-tuples of integers $(x_1, x_2, \cdots, x_n)$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n$ and d divides $x_1 + x_2 + \cdots + x_n$. Prove that exactly half of the n-tuples in S has the property that $x_n = n$.

3. Let $ABCDEF$ be a convex hexagon with area S. Prove that

$$|AC| \cdot (|DB| + |BF| - |FD|) + |CE| \cdot (|FD| + |DB| - |BF|) + |EA| \cdot (|BF| + |FD| - |DB|) \geq 2\sqrt{3}S.$$

[4] Just to be clear, all interior angles of the convex polygon must be strictly less than 180^o, i.e. no three vertices are collinear.

2009 Mock Olympiad 13: (IMO Mock)
Time: 4.5 Hours

1. Find all positive integers $n \geq 3$ such that there exists a convex polygon with n vertices on the Cartesian coordinate plane such that the side lengths are all odd integers and every vertex has integer coordinates. [5]

Solution: The answer is all $n \geq 4$ even.

Let $(x_1, y_1), \cdots, (x_n, y_n)$ be the coordinates of the vertices in order around the polygon. Suppose n is odd. Since the side lengths of each polygon is an odd integer, either x_i, x_{i+1} have different parities or y_i, y_{i+1} have different parities (but not both). (We are using the notation that $x_i = x_{n+i}$.) We call a pair (x, y) red if x, y are the same parity and blue otherwise. Then (x_i, y_i) must have different colours than (x_{i+1}, y_{i+1}). Since n is odd, this is impossible.

To prove that n is even works, note that $(2k + 1, 2k^2 + 2k, 2k^2 + 2k + 1)$ is a primitive Pythagorean triple. Let $a_k = 2k+1$ and $b_k = 2k^2+2k$. Then a triangle with with leg lengths a_k, b_k will have a hypotenuse with odd integral length. Also note that $a_k/b_k < 1$ and b_k/a_k is a strictly increasing function in variable $k \geq 1$.

If $n = 4$, then we can simply choose the vertices of a square to be $(0, 0), (1, 0), (1, 1), (0, 1)$. Let $n = 2m$. Let $X_1 = (4, 3)$ and $X_i = (a_i, b_i)$ for $i = 2, \cdots, 2m-3$. Let $S = 4+a_2+a_3+\cdots+a_{2m-3}$, i.e. the sum of the x-coordinates of these points and $T = 3 + b_2 + b_3 + \cdots + b_{2m-3}$, the sum of the y-coordinates of these points. Since $b_i - a_i > 1$ for $i \geq 2$, we have that $S < T$. Since a_1 is even and b_i's are odd, we have that S is even and T is odd. Let $P_0 = (0, 0)$ and $P_{i+1} = P_i + X_{2m-3-i}$ for $i = 1, 2, \cdots, 2m-4$, $P_{2m-2} = (T, T)$ and $P_{2m-1} = (T, 0)$. I claim that $P_0, \cdots, P_{2m-1}$ are the vertices of a convex n-gon with odd integral sides. Since T is odd, the segments $P_{2m-2}P_{2m-1}$ and $P_{2m-1}P_0$ have odd integral length. By choice of (a_i, b_i), P_iP_{i+1} have odd lengths for $i = 1, 2, \cdots, 2m-4$. Finally, note that $P_{2m-2} = (T - S, T)$. Since S is even and T is odd, $T - S$ is odd, therefore, $P_{2m-2}P_{2m-1}$ has odd length. It remains to show that the polygon is convex. The only points on the line $y = x$ are P_0 and P_{2m-2}, this is because P_iP_{i+1} has slope greater than 1 for all $i = 0, 1, \cdots, 2m-4$. Therefore, the y-coordinate of P_{2m-3} is at least $2m - 3 > 1$ more than the x-coordinate. Since $P_{2m-2} = P_{2m-3} + (4, 3)$, the point P_{2m-2} is also above the line $y = x$. The point $P_{2m-1} = (T, 0)$ is clearly below the line $y = x$. To show that it is convex, note that the slopes P_iP_{i+1} for $i = 0, 1, \cdots, 2m-4$ are greater than 1 and in strictly decreasing order and $P_{2m-3}P_{2m-2}$ has slope less than 1. Therefore, the points $P_0, \cdots, P_{2m-1}$ are points lying on or above $y = x$ and the slopes of the lines P_iP_{i+1} are in strictly decreasing order. Therefore, the polygon is convex. $\square$

Source: China Team Selection Test 2008

[5] Just to be clear, all interior angles of the convex polygon must be strictly less than 180^o, i.e. no three vertices are collinear.

2. Let d, n be positive integers such that d divides n. Let S be the set of all (ordered) n-tuples of integers $(x_1, x_2, \cdots, x_n)$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n$ and d divides $x_1 + x_2 + \cdots + x_n$. Prove that exactly half of the n-tuples in S has the property that $x_n = n$.

Solution: We will create a bijection between the subset of S with the property $x_n = n$ and the subset of S without the property $x_n = n$.

For each $(x_1, \cdots, x_n)$, we define a part of this element to be each component $x_i > 0$. Hence, x_n is the largest part. Let X be the subset of S with $x_n = n$. Let X_1 be the subset of X with less than n parts. Let X_2 be the subset of X with exactly n parts.

Each element of S can be represented as follows: Given an n by n grid, where the i^{th} row has its first x_i columns coloured red. Note that such a colouring corresponds to an element in S if and only if each row has at least as many coloured squares as any row above it. Also note that each column has at least as many coloured squares as any column to the right of it. Therefore, if we reflect this about the diagonal containing the lower-left, the resulting colouring still corresponds to an element of S. Hence, there is a bijection of X_1: elements of S with largest part n with less than n parts with elements of S with largest part less than n with exactly n parts. Let this latter set be Y_1. Hence, $|X_1| = |Y_1|$.

Consider X_2, the subset of S with largest part n and with exactly n parts. Given $U \in X_2$, remove all parts equal to n and pre-append 0's to the beginning of U. Formally, if U contains k $n's$ for some $k > 1$, then erase $x_n, x_{n-1}, \cdots, x_{n-k+1}$, then replace x_i with x_{i-k} for $i \in \{k+1, \cdots, n\}$ and set $x_j = 0$ for all $j \in \{1, \cdots, k\}$. This results in an element of S with largest part less than n and number of parts less than n, since $d|n$ and the resulting element has sum which is a multiple of n less than the original element, which means the property that d divides the sum of the elements is preserved. Call elements of this form Y_2. Conversely, given an element in Y_2, we can erase the 0's and append n's to the element to recover the corresponding element of X_2. Therefore, there is a bijection between X_2 and Y_2. Therefore, $|X_2| = |Y_2|$. All elements of S with largest part less than n is in $Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$. All elements in S with largest part n is exactly $X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Furthermore, X_1, X_2, Y_1, Y_2 are pairwise non-intersecting. Therefore, $|X_1| + |X_2| = |Y_1| + |Y_2|$ meaning the number of elements in S with largest part n is equal to the number of elements in S with largest part not equal to n. This solves the problem. $\square$

Source: China Team Selection Test 2006

3. Let $ABCDEF$ be a convex hexagon with area S. Prove that

$$|AC|\cdot(|DB|+|BF|-|FD|)+|CE|\cdot(|FD|+|DB|-|BF|)+|EA|\cdot(|BF|+|FD|-|DB|) \geq 2\sqrt{3}S.$$

Solution: Let O be any point in the interior of $ABCDEF$. Note that $|AC|\cdot|BO| \geq 2[ABCO]$ (where $[\cdots]$ denotes area). Similarly, $|CE|\cdot|DO| \geq 2[CDEO]$ and $|EA|\cdot|FO| \geq 2[EDAO]$. Since $[ABCO]+[CDEO]+[EFAO] = [ABCDEF]$, we have that

$$|AC|\cdot|BO| + |CE|\cdot|DO| + |EA|\cdot|FO| \geq 2[ABCDEF] = 2S.$$

Let the incircle of ΔBDF touch DF, FB, BD at X, Y, Z, respectively. Note that

$$|BY| = \frac{|BD|+|BF|-|DF|}{2}, |DZ| = \frac{|BD|+|DF|-|BF|}{2}, |FX| = \frac{|BF|+|DF|-|BD|}{2}.$$

Hence, it suffices to show that there exists a point O such that

$$\frac{|BY|}{|BO|} \geq \frac{\sqrt{3}}{2}, \frac{|CZ|}{|CO|} \geq \frac{\sqrt{3}}{2}, \frac{|AX|}{|AO|} \geq \frac{\sqrt{3}}{2},$$

since this would imply LHS of inequality equals $2(|AC|\cdot|BY|+|CE|\cdot|DZ|+|EA|\cdot|FX|) \geq \sqrt{3}(|AC|\cdot|BO|+|CE|\cdot|DO|+|EA|\cdot|FO|) \geq 2\sqrt{3}S$, which proves the problem.

Let r be the real number such that be the three circles, centred at B, D, F, respectively, which are the images of a dilation of ratio r of the circles with centres B, D, F respectively, have exactly one point O in common. It suffices to show that $r \leq \frac{2}{\sqrt{3}}$. Suppose $r > \frac{2}{\sqrt{3}}$. At least one angle $\angle BOD, \angle DOF, \angle FOB$ is at least 120^o. Suppose it is $\angle BOD$. Then

$$\cos\angle BOD = \frac{|OB|^2+|OD|^2-|BD|^2}{2\cdot|OB|\cdot|OD|} > \frac{\frac{4}{3}|BZ|^2+\frac{4}{3}|DZ|^2-(|BZ|+|DZ|)^2}{\frac{8}{3}|BZ||DZ|} \geq \frac{-1}{2},$$

by AM-GM inequality, contradicting $\angle BOD > 120^o$. Therefore, $r \leq \frac{2}{\sqrt{3}}$. This completes the problem.

Source: International Zhautykov Olympiad 2009

2009 Mock Olympiad 14: (IMO Mock)
Time: 4.5 Hours

1. Let ABC be a triangle with circumcircle Γ. Let γ be a circle tangent to side BC at a point Q and internally tangent to Γ at a point P on the arc BC on Γ, not containing A. Let J be the centre of γ. Suppose J lies on the internal angle bisector of $\angle BAC$. Prove that J lies on the internal angle bisector of $\angle PAQ$.

2. Let f, g, b be polynomials with real coefficients such that f, g are in one variable and b is in two variables. Suppose that

$$f(x) - f(y) = b(x, y)(g(x) - g(y))$$

for all $x, y \in \mathbb{R}$. Prove that there exists a polynomial h with real coefficients such that $f(x) = h(g(x))$ for all $x \in \mathbb{R}$.

3. Prove that for any odd prime number p, the number of positive integers n such that p divides $n! + 1$ is less than or equal to $cp^{\frac{2}{3}}$, where c is a constant independent of p.

2009 Mock Olympiad 14: (IMO Mock)
Time: 4.5 Hours

1. Let ABC be a triangle with circumcircle Γ. Let γ be a circle tangent to side BC at a point Q and internally tangent to Γ at a point P on the arc BC on Γ, not containing A. Let J be the centre of γ. Suppose J lies on the internal angle bisector of $\angle BAC$. Prove that J lies on the internal angle bisector of $\angle PAQ$.

   Solution: Let M be the midpoint of the arc BC on Γ not containing A. I hope you guys know by now that this implies A, J, M are collinear. We will now prove that the quadrilateral $APJQ$ is cyclic. Let ϕ : be the dilation that maps γ to Γ with the centre of dilation at P. This is possible since γ is internally tangent to Γ. Consider the point where ϕ maps Q, and call it X. Then the line tangent to Γ at X is parallel to BC, since γ is tangent to BC at Q. Therefore, X is on the midpoint of the arc BC containing A. Hence, X is diametrically opposite M.

   Therefore, $\angle AJQ = \angle AMX = \angle APX = \angle APQ$. This implies $APJQ$ cyclic. Since $\angle JPQ = \angle JQP$, we have that $\angle JAQ = \angle JAP$, which implies J lies on the angle bisector of $\angle QAP$, as desired. $\square$

   Source: Japan Mathematical Olympiad 2009 Final Round

2. Let f, g, b be polynomials with real coefficients such that f, g are in one variable and b is in two variables. Suppose that

$$f(x) - f(y) = b(x, y)(g(x) - g(y))$$

   for all $x, y \in \mathbb{R}$. Prove that there exists a polynomial h with real coefficients such that $f(x) = h(g(x))$ for all $x \in \mathbb{R}$.

   Solution: If $\deg g = 0$, then $g(x) - g(y) = 0$ for all $x, y \in \mathbb{R}$. Therefore, $f(x) - f(y) = 0$ for all $x, y \in \mathbb{R}$. Therefore, f is a constant. Setting h to be this constant gives the desired function. We will henceforth assume that $\deg g > 0$. Note that by interchanging x, y, we can conclude that $b(x, y) = b(y, x)$. Therefore, $\deg_x(b) = \deg_y(b)$.

   Clearly, $\deg f - \deg g \geq 0$. We will solve the problem by induction on $\deg f - \deg g$. If $\deg f - \deg g = 0$, then $\deg_x(b) = \deg_y(b) = 0$. Hence, $b(x, y)$ is a constant polynomial. Let c be this constant. Then $f(x) - f(y) = c(g(x) - g(y))$ for all $x, y \in \mathbb{R}$. Substituting $y = 0$ yields $f(x) = cg(x)$. Let $h(x) = cx$. Then $h(g(x)) = c \cdot g(x) = f(x)$ for all $x \in \mathbb{R}$. Therefore, we have found the desired h.

   Now suppose this holds for all f, g such that $\deg f - \deg g \in \{0, 1, \cdots, r-1\}$. Suppose $\deg f - \deg g = r$. Let $F(x) = f(x) - f(0)$ and $G(x) = g(x) - g(0)$ and $B(x) = b(x, 0)$. Then we have that $F(x) = B(x)G(x)$ and $F(x) - F(y) = b(x, y)(G(x) - G(y))$. Note that

$F(x)-F(y)=B(x)G(x)-B(y)G(y)=B(x)G(x)-B(x)G(y)+B(x)G(y)-B(y)G(y)=B(x)(G(x)-G(y))+G(y)(B(x)-B(y))$. Since $G(x)-G(y)$ divides $F(x)-F(y)$ and $G(x)-G(y)$ is relatively prime to $G(y)$, we conclude that $G(x)-G(y)$ divides $B(x)-B(y)$, i.e. there exists a two variable polynomial $a(x,y)$ with real coefficients such that

$$B(x)-B(y)=a(x,y)(G(x)-G(y)).$$

Since $\deg g>0$, $\deg G>0$ and subsequently, $\deg B-\deg G<\deg F-\deg G$. By induction hypothesis, we have that $B(x)=J(G(x))$ for some polynomial J with real coefficients. Therefore, $F(x)=J(G(x))G(x))$. Let $K(x)=xJ(x)$. Then $F(x)=K(G(x))$. Then $f(x)=F(x)+f(0)=K(G(x))+f(0)$. Setting $H(x)=K(x)+f(0)$ gives us $f(x)=H(G(x))$, as desired. This completes the proof. $\square$

Source: IMO 1992 Shortlist

3. Prove that for any odd prime number p, the number of positive integers n such that p divides $n!+1$ is less than or equal to $cp^{\frac{2}{3}}$, where c is a constant independent of p.

Solution: Clearly all possible integers n are in the range $1\le n<p$. Let $a_1,a_2,\cdots,a_r$ be all of the integers in this range such that p divides $a_i!+1$ with $a_1<a_2<\cdots<a_r$. We need to prove that $r<cp^{2/3}$ for some constant c independent of p. For each $j\in\{1,2,\cdots,p-2\}$, let

$$x_j=|\{i\in\{1,2,\cdots,r-1\}\text{ such that} a_{i+1}-a_i=j|.$$

Then

$$x_1+x_2+\cdots+x_{p-2}=r-1.$$

Furthermore,

$$x_1+2x_2+3x_3+\cdots+(p-2)x_{p-2}=a_r-a_1=p-2.$$

Since $a_1=1$ and $a_r=p-1$ since $(p-1)!=-1 \mod p$ by Wilson's Theorem. I also claim that $x_j\le j$ for each j. Suppose $a_{i+1}-a_i=j$. Since $p|a_i!+1$ and $p|a_{i+1}!+1$, we have that p divides $a_{i+1}!-a_i!$. Since a_i is relatively prime with p, we have that p divides $(a_i+1)(a_i+2)\cdots(a_i+(a_{i+1}-a_i))-1$. But the polynomial $(x+1)(x+2)\cdots(x+(a_{i+1}-a_i))-1=(x+1)(x+2)\cdots(x+j)-1$ has at most j solutions modulo p. Therefore, $x_j\le j$.

We want to maximize $S=x_1+x_2+\cdots+x_{p-2}$ if $x_1,\cdots,x_{p-2}$ are non-negative real numbers such that $x_1+2x_2+\cdots+(p-2)x_{p-2}=p-2$ and $0\le x_i\le i$. Note that S is maximized when the terms with the smallest indices x_i are maximized. Let m be the largest index such that $1^2+2^2+\cdots+m^2<p-2$. Then $p-2=1^2+2^2+\cdots+m^2+(m+1)t$ for some $0\le t<m+1$. Therefore,

$$\frac{m(m+1)(2m+1)}{6}<p-2\Rightarrow\frac{m^3}{3}<p\Rightarrow m<(3p)^{1/3}.$$

Then, the maximum possible value of r is $S+1=1+2+\cdots+m+t+1=\frac{m(m+1)}{2}+t<\frac{(m+1)(m+2)+2}{2}<\frac{8m^2}{2}=4m^2=4(3p)^{2/3}$. Setting $c=4\cdot 3^{2/3}$ gives the desired result. $\square$

Source: China Team Selection Test 2009

2009 Mock Olympiad 15: (IMO Mock)
Time: 4.5 Hours

1. Find the maximum possible finite number of roots to the equation

$$|x-a_1|+\ldots+|x-a_{50}|=|x-b_1|+\ldots+|x-b_{50}|,$$

where $a_1,\ a_2,\ \ldots,a_{50},\ b_1,\ldots,\ b_{50}$ are distinct reals.

2. Let m,n be positive integers and $f(x)$ be a degree polynomial of degree n such that each coefficient of the polynomial is odd. Suppose f has a factor of $(x-1)^m$. If $m\geq 2^k(k\geq 2,k\in N)$, prove that $n\geq 2^{k+1}-1$.

3. Let ABC be an acute triangle, let M,N be the midpoints of minor arcs $\widehat{CA},\widehat{AB}$ of the circumcircle of triangle ABC, point D is the midpoint of segment MN, point G lies on minor arc $\widehat{BC}$. Denote by I,I_1,I_2 the incenters of triangle ABC,ABG,ACG respectively. Let P be the second intersection of the circumcircle of triangle GI_1I_2 with the circumcircle of triangle ABC. Prove that the points D,I,P are collinear.

2009 Mock Olympiad 15: (IMO Mock)
Time: 4.5 Hours

1. Find the maximum possible finite number of roots of the equation

$$|x - a_1| + \ldots + |x - a_{50}| = |x - b_1| + \ldots + |x - b_{50}|,$$

where $a_1, a_2, \ldots, a_{50}, b_1, \ldots, b_{50}$ are distinct reals.

Solution: The answer is 49.

Let

$$f(x) = \sum_{i=1}^{50} |x - a_i| - \sum_{i=1}^{50} |x - b_i|.$$

Let $\{c_1, c_2, \cdots, c_{100}\} = \{a_1, a_2, \cdots, a_{50}\} \cup \{b_1, b_2, \cdots, b_{50}\}$ with $c_1 < c_2 < \cdots < c_{100}$. For each $i \in \{1, 2, \cdots, 100\}$, let $\epsilon_i = 1$ if $c_i = a_j$ for some j and -1 if $c_i = b_j$ for some j. Note that $f(x)$ is continuous and consists of 101 piece-wise linear functions.

If $x < c_1$, then

$$f(x) = \sum_{i=1}^{50}(a_i - x) - \sum_{i=1}^{50}(b_i - x) = \sum_{i=1}^{50}(a_i - b_i),$$

which is a constant. Similarly, if $x > c_{100}$, then

$$f(x) = \sum_{i=1}^{50}(b_i - a_i).$$

This constant cannot be zero, for otherwise $f(x)$ would have infinitely many roots. Therefore, we have that $f(\infty) + f(-\infty) = 0$.

If $c_j \leq x < c_{j+1}$, we have that

$$f(x) = \sum_{i=1}^{100} \epsilon_i |x - c_i| = \sum_{i=1}^{j} \epsilon_i (c_i - x) + \sum_{i=j+1}^{100} \epsilon (x - c_i)$$

$$= (\epsilon_1 + \cdots + \epsilon_j - \epsilon_{j+1} - \cdots - \epsilon_{100})x + (-\epsilon_1 c_1 - \cdots - \epsilon_j c_j + \epsilon_{j+1} c_{j+1} + \cdots + \epsilon_{100} c_{100}).$$

Let m_j be the coefficient of x of this expression and b_j be the constant. Then m_j is the slope of the function in the interval $[c_j, c_{j+1}]$. Note that since $\epsilon_i \in \{-1, 1\}$, m_j is an even integer and m_j and m_{j+1} differ by 2. If $m_j = 0$, then $b_j \neq 0$ since otherwise, this would imply that every number in the interval $[c_j, c_{j+1}]$ is a root of f, which is impossible since f has finitely many roots. Since $m_j < 0$ and $m_{j+1} > 0$ is impossible (since m_j is even and $|m_{j+1} - m_j| = 2$, we have that no adjacent intervals of the form $[c_j, c_{j+1}]$ can have roots. There are 99 intervals between $[c_1, c_{100}]$. Therefore, f has at most 50 roots. But if f has exactly 50 roots, then the intervals that have roots are $[c_1, c_2], [c_3, c_4], \cdots, [c_{99}, c_{100}]$. Furthermore, by a parity argument, $m_1, m_3, m_5, \cdots, m_{99}$ must have alternating signs. This means m_1, m_{99} have opposite signs. This is impossible since $f(\infty)$ and $f(-\infty)$ have opposite signs. Therefore, f has at most 49 roots.

Now, we construct an example. Let $\{a_i\}_{i=1}^{50} = \{1, 4, 5, 8, 9, \cdots, 93, 96, 97, 100 - 0.5\}$ and $\{b_i\}_{i=1}^{50} = \{2, 3, 6, 7, \cdots, 94, 95, 98, 99\}$. Then $f(-\infty) = -1/2$ and $f(\infty) = 1/2$. (Note that we cannot set $a_{50} = 100$ for otherwise, $f(-\infty) = 0$, which is not allowed. So we tweak it and set $a_{50} = 99.5$.) Define m_j, b_j as previously for these specific values of a_i and b_i. Therefore,

$$\{m_j\}_{j=1}^{100} = \{2, 0, -2, 0, 2, 0, -2, 0, \cdots, 2, 0, -2, 0\}$$

and

$$\{b_j\}_{j=1}^{100} = \{-2.5, 1.5, 7.5, -0.5, -10.5, 1.5, 15.5, -0.5, \cdots, -194.5, 1.5, 199.5, 0.5)\}.$$

(The last term deviates from the pattern due to the tweak of a_{50}.) Then we have that $f(c_1) = -0.5, f(c_2) = f(c_3) = 1.5, f(c_4) = f(c_5) = -0.5, \cdots, f(c_{98}) = f(c_{99}) = 1.5, f(c_{100}) = 0.5$. Therefore, by Intermediate Value Theorem, there is a root in each of the intervals $[c_1, c_2], [c_3, c_4], \cdots, [c_{97}, c_{98}]$. Therefore, f has 49 roots. $\square$

Source: Russian Mathematical Olympiad 2005 - Grade 11

2. Let m, n be positive integers and $f(x)$ be a degree polynomial of degree n such that each coefficient of the polynomial is odd. Suppose f has a factor of $(x-1)^m$. If $m \geq 2^k (k \geq 2, k \in N)$, prove that $n \geq 2^{k+1} - 1$.

   Solution: Given a polynomial f with integer coefficients, let $\overline{f}$ be the polynomial whose coefficients are reduced modulo 2. Since $f(x)$ has coefficients in $\{1, -1\}$, we have that $\overline{f}(x) = x^n + x^{n-1} + \cdots + x + 1$. Since $f(x) = (x-1)^{2^k} g(x)$ for some polynomial g with integer coefficients, we have that

$$x^n + x^{n-1} + \cdots + x + 1 = (x-1)^{2^k} \overline{g}(x).$$

   We will leave to the reader to prove that $\binom{2^k}{a}$ is even for $1 \leq a \leq 2^k - 1$. Therefore,

$$x^n + x^{n-1} + \cdots + x + 1 = (x^{2^k} - 1)\overline{g}(x).$$

   Suppose $\deg \overline{g} \leq 2^k - 2$. Then the coefficient of $x^{2^k - 1}$ of $(x^{2^k} - 1)\overline{g}(x)$ would be zero. But it must be one since the left-hand side is equal to $x^n + x^{n-1} + \cdots + x + 1$. Therefore, $\deg \overline{g} \geq 2^k - 1$, which means $\deg g \geq 2^k - 1$. Hence, $\deg f \geq 2^k + 2^k - 1 = 2^{k+1} - 1$, as desired. $\square$

   Source: China Team Selection Test 2009

3. Let ABC be an acute triangle, let M, N be the midpoints of minor arcs $\widehat{CA}, \widehat{AB}$ of the circumcircle of triangle ABC, point D is the midpoint of segment MN, point G lies on minor arc $\widehat{BC}$. Denote by I, I_1, I_2 the incenters of triangle ABC, ABG, ACG respectively. Let P be the second intersection of the circumcircle of triangle GI_1I_2 with the circumcircle of triangle ABC. Prove that three points D, I, P are collinear.

Solution: We claim that

$$\frac{PN}{NA} = \frac{PM}{MA}.$$

Note that G, I_1, N are collinear and G, I_2, M are collinear. Also recall that $NA = NI = NB = NI_1$ and $MA = MI = MC = MI_2$. Since PI_1I_2G is cyclic, we have that $\angle PI_1G = \angle PI_2G$. We have that $\angle NI_1P = \angle MI_2P$. Also, $\angle PNI_1 = \angle PNG = \angle PMG = \angle PMI_2$. Therefore, $\Delta PI_1N \sim \Delta PI_2M$. Since $NI_1 = NA$ and $MI_2 = MA$, we have that

$$\frac{PN}{NA} = \frac{PM}{MA}.$$

This proves our claim.

Note that since $NA/NM = NI/IM = NP/PM$, then the circumcircle of ΔAIP is an Apollonius circle for M, N. We leave for the reader to show that the power of the point of D with respect to the circumcircle of ΔAIP is DN^2. Let DI intersect the circumcircle of ΔABC at A' and P', where A' is on the same side of line MN as A. Since D is the midpoint of MN, we have that A' is the reflection of A about the perpendicular bisector of MN. Therefore, $\angle NP'I = \angle NMQ = \angle ANM = \angle INM$ since $ANIM$ is a rhombus since $NA = NI$ and $MA = MI$. Therefore, $DN^2 = DI \cdot DP'$. The former is the power of a point to the circumcircle of ΔAIP. Therefore, $P = P'$. This proves that D, I, P are collinear. $\square$