SKOLIAD No. 116

Lily Yen and Mogens Hansen

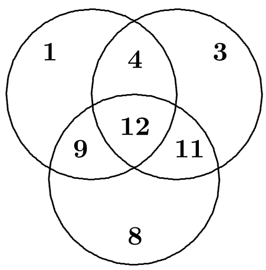
Please send your solutions to problems in this Skoliad by ***1 October, 2009***. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

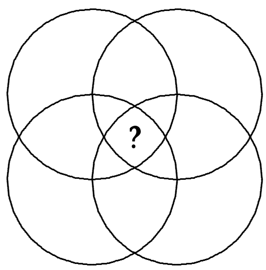
Our contest this month is the Final Round of the Swedish Junior High School Mathematics Contest 2007/2008. Our thanks go to Paul Vaderlind, Stockholm University, Stockholm, Sweden for providing us with this contest and for permission to use it. We also thank Jean-Marc Terrier of the University of Montreal for translating this contest from English into French.

Swedish Junior High School Mathematics Contest Final Round, 2007/2008

3 hours allowed

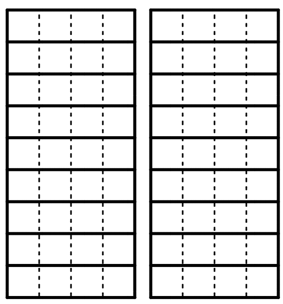
1. Values are assigned to a number of circles, and these values are written in the circles. When two or more circles overlap, the sum of the values of the overlapping circles is written in the common region. In the example on the left below, the three circles have been assigned the values 1, 3, and 8. Where the circle with value 1 overlaps the circle with value 3 we write 4 $(= 1 + 3)$. In the region in the middle, we add all three values and write 12.



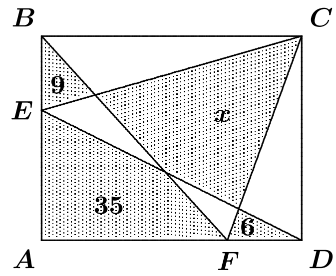


In the figure on the right above are four circles and, thus, thirteen regions. Find the number in the middle if the sum of all thirteen numbers is 294.

2. This is the 20th edition of the Swedish Junior High School Mathematics Contest. The first qualification round was held in the fall of 1988, and this year's final is held in 2008. That is twenty-one calendar years, 1988–2008, but the table below has room for only eighteen of them. Which three must be omitted if the digit sum in every row and every column must be divisible by 9? (Two solutions exist.)



3. The line segments DE, CE, BF, and CF divide the rectangle $ABCD$ into a number of smaller regions. Four of these, two triangles and two quadrilaterals, are shaded in the figure at right. The areas of the four shaded regions are 9, 35, 6, and x (see the figure). Determine the value of x.



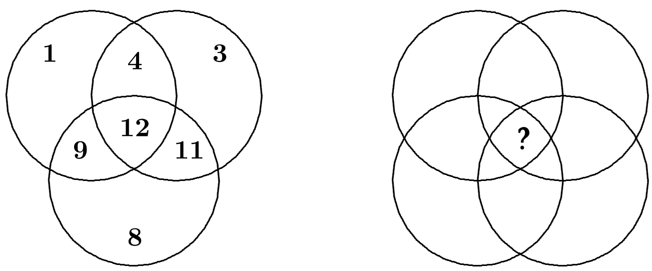
4. A goody bag contains a two-digit number of goodies. Lisa adds the two digits and then removes as many goodies as the sum yields. Lisa repeats this procedure until the number of goodies left is a single digit number larger than zero. Find this single digit number.

5. In how many ways can the list $[1, 2, 3, 4, 5, 6]$ be permuted if the product of neighbouring numbers must always be even?

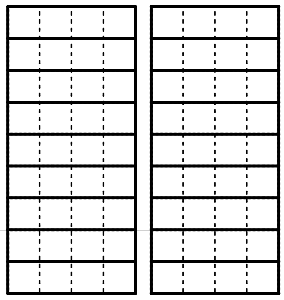
6. The digits of a five-digit number are $abcde$. Prove that $abcde$ is divisible by 7 if and only if the number $abcd - 2 \cdot e$ is divisible by 7.

Concours suédois de mathématiques précollégiales
Ronde finale, 2007/2008
Durée : 3 heures

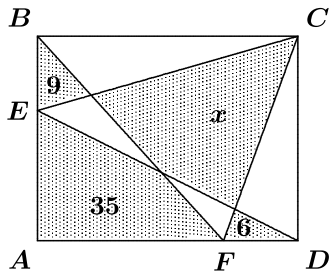
1. On attribue des valeurs à un certain nombre de cercles, et on les écrit dans les cercles correspondants. Quand plusieurs cercles se coupent, la somme des valeurs des cercles qui se recoupent est inscrite dans la région commune. Dans l'exemple ci-dessous à gauche, les trois cercles ont comme valeurs 1, 3 et 8. Là où le cercle de valeur 1 recoupe le cercle de valeur 3, on écrit 4 $(= 1 + 3)$. Dans la région du milieu, on additionne les trois valeurs et on écrit 12.



Dans la figure de droite ci-dessus, on a quatre cercles avec treize régions. Trouver le nombre dans le milieu si la somme des treize nombres est 294.



2. Ceci est la 20e édition du Concours suédois de mathématiques précollégiales. La première ronde de qualification a eu lieu en automne 1988, et la finale de cette année a lieu en 2008. Cela fait vingt-et-une années calendrier, 1988–2008, mais la table ci-contre n'a de place que pour dix-huit d'entre elles. Quelles sont les trois qui doivent être omises si la somme des chiffres dans chaque ligne et dans chaque colonne doit être divisible par 9 ? (Il y a deux solutions.)



3. Les segments DE, CE, BF et CF divisent le rectangle $ABCD$ en un certain nombre de régions plus petites. Dans la figure de droite, quatre d'entre elles sont ombrées, deux triangles et deux quadrilatères. Les aires des quatre régions ombrées sont 9, 35, 6 et x (voir la figure). Déterminer la valeur de x.

4. Le nombre de surprises contenues dans un sac à surprises comporte deux chiffres. Lise retire du sac autant de surprises que le total de son addition des deux chiffres. Elle répète la même procédure jusqu'à ce que le nombre de surprises restantes soit un nombre positif d'un seul chiffre. Quel est ce nombre ?

5. De combien de manières la liste $[1, 2, 3, 4, 5, 6]$ peut-elle être permutée si le produit de nombres voisins doit toujours être pair ?

6. Un nombre comporte les cinq chiffres $abcde$. Montrer que $abcde$ est divisible par 7 si et seulement si le nombre $abcd - 2 \cdot e$ est divisible par 7.

Next we give solutions to the County Competition run by the Croatian Mathematical Society 2007 given at [2008 : 195-196].

1. Find all integer solutions to the equation $x^2 + 11^2 = y^2$.

Solution by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.

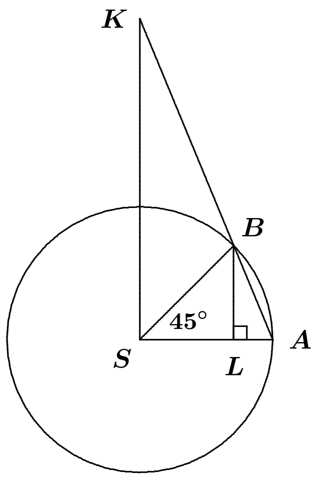
The given equation is equivalent to $(y + x)(y - x) = 11^2$.

Hence, $y + x$ and $y - x$ are divisors of 11^2, namely ± 1, ± 11, and ± 121. If $y + x = \pm 1$ and $y - x = \pm 121$, then $y = \pm 61$ and $x = \mp 60$. If $y + x = \pm 11$ and $y - x = \pm 11$, then $y = \pm 11$ and $x = 0$. If $y + x = \pm 121$ and $y - x = \pm 1$, then $y = \pm 61$ and $x = \pm 60$. Hence, the solutions for (x, y) are $(-60, 61)$, $(60, -61)$, $(0, 11)$, $(0, -11)$, $(60, 61)$, and $(-60, -61)$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

2. In a circle with centre S and radius $r = 2$, two radii SA and SB are drawn. The angle between them is $45°$. Let K be the intersection of the line AB and the perpendicular to line AS through point S. Let L be the foot of the altitude from vertex B in ΔABS. Determine the area of trapezoid $SKBL$.

Solution by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.



First, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$, so we have

$$-1 = \tan 135° = \frac{2\tan(135°/2)}{1-\tan^2(135°/2)},$$

and thus $\tan\dfrac{135°}{2} = 1+\sqrt{2}$.

Since $\angle SAB = \dfrac{135°}{2}$, we then have $SK = 2\tan\angle SAB = 2+2\sqrt{2}$. Hence, the area of $\triangle SAK$ is $\frac{1}{2}SA\cdot SK = 2+2\sqrt{2}$.

Also, $SL = BL = SB\cdot\sin 45° = \sqrt{2}$, so the area of $\triangle SBL$ is $\frac{1}{2}SL\cdot BL = 1$ and the area of $\triangle SBA$ is $\frac{1}{2}SA\cdot BL = \sqrt{2}$. The area of $\triangle LBA$ is the difference of the areas of $\triangle SBA$ and $\triangle SBL$, which is $\sqrt{2}-1$. Therefore, the area of trapezoid $SKBL$ is $\left(2+2\sqrt{2}\right)-\left(\sqrt{2}-1\right) = 3+\sqrt{2}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

The use of trigonometry can be avoided by noting that $\triangle ALB$ is similar to $\triangle ASK$.

3. Let a, b, and c be given nonzero real numbers. Find x, y, and z if

$$\frac{ay+bx}{xy} = \frac{bz+cy}{yz} = \frac{cx+az}{zx} = \frac{4a^2+4b^2+4c^2}{x^2+y^2+z^2}.$$

Solution by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.

We have that $\frac{a}{x}+\frac{b}{y} = \frac{b}{y}+\frac{c}{z} = \frac{c}{z}+\frac{a}{x}$, which implies that $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$. Let k be this common value. Then $a = kx$, $b = ky$, and $c = kz$. Substituting into the given equation yields that $2k = \dfrac{4k^2x^2+4k^2y^2+4k^2z^2}{x^2+y^2+z^2} = 4k^2$ hence $k = 0$ or $k = \frac{1}{2}$. If $k = 0$, then $a = b = c = 0$, contradicting the requirement that a, b, and c are nonzero. If $k = \frac{1}{2}$, then $x = 2a$, $y = 2b$, and $z = 2c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

4. Let a and b be positive real numbers such that $a > b$ and $ab = 1$. Prove the inequality

$$\frac{a-b}{a^2+b^2} \;\leq\; \frac{\sqrt{2}}{4}\,.$$

Determine $a+b$ if equality holds.

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

The given inequality is successively equivalent to

$$\begin{aligned}
\frac{a-b}{a^2+b^2} &\leq \frac{\sqrt{2}}{4}\,; \\
\frac{(a-b)^2}{(a^2+b^2)^2} &\leq \frac{1}{8}\,; \\
8(a-b)^2 &\leq \left(a^2+b^2\right)^2\,; \\
0 &\leq \left(a^2+b^2\right)^2 \;-\; 8\left(a^2+b^2\right) \;+\; 16ab\,.
\end{aligned}$$

Since $ab = 1$, the last inequality is equivalent to

$$0 \;\leq\; \left(a^2+b^2\right)^2 \;-\; 8\left(a^2+b^2\right) \;+\; 16 \;=\; \left(a^2+b^2-4\right)^2\,,$$

which is obviously true. Moreover, equality holds if and only if

$$0 \;=\; a^2+2+b^2-6 \;=\; a^2+2ab+b^2-6 \;=\; (a+b)^2-6\,,$$

that is, if and only if $a+b = \sqrt{6}$.

5. The ratio between the lengths of two sides of a rectangle is $12 : 5$. The diagonals divide the rectangle into four triangles. Circles are inscribed in two of them having a common side. Let r_1 and r_2 be their radii. Find the ratio $r_1 : r_2$.

Solution by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.

Recall that the radius of the circle inscribed in a triangle with area F and perimeter P is $2F/P$.

Let $ABCD$ be the rectangle and assume without loss of generality that $AB = 5$ and $BC = 12$. By the Pythagorean Theorem, each diagonal of $ABCD$ has length 13. Divide the rectangle into four triangles each with area $\dfrac{5\cdot 12}{4} = 15$. The perimeters of adjacent triangles are then $\dfrac{13}{2}+\dfrac{13}{2}+5 = 18$ and $\dfrac{13}{2}+\dfrac{13}{2}+12 = 25$. Therefore, the radii of the circles inscribed in these triangles are $r_1 = \dfrac{2\cdot 15}{18}$ and $r_2 = \dfrac{2\cdot 15}{25}$, so the ratio $r_1 : r_2$ is $25 : 18$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff ~~members are Monika Khbeis (Ascension of Our Lord~~ Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Please send your solutions to the problems in this edition by **15 July 2009**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M388. *Proposed by Kyle Sampson, St. John's, NL.*

A sequence is generated by listing (from smallest to largest) for each positive integer n the multiples of n up to and including n^2. Thus, the sequence begins 1, 2, 4, 3, 6, 9, 4, 8, 12, 16, 5, 10, 15, 20, 25, 6, 12, $\dots$. Determine the 2009^{th} term in the sequence.

M389. *Proposed by Lino Demasi, student, Simon Fraser University, Burnaby, BC.*

There are 2009 students and each has a card with a different positive integer on it. If the sum of the numbers on these cards is 2020049, what are the possible values for the median of the numbers on the cards?

M390. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

A Pythagorean triangle is a right-angled triangle with all three sides of integer length. Let a and b be the legs of a Pythagorean triangle and let h be the altitude to the hypotenuse. Determine all such triangles for which

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1\,.$$

M391. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

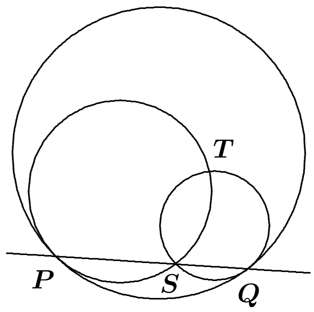
Determine all pairs (a, b) of positive integers for which both $\frac{a+1}{b}$ and $\frac{b+2}{a}$ are positive integers.

M392. *Proposed by the Mayhem Staff.*

Determine, with justification, the fraction $\frac{p}{q}$, where p and q are positive integers and $q < 1000$, that is closest to, but not equal to, $\frac{19}{72}$.

M393. *Proposed by the Mayhem Staff.*

Inside a large circle of radius r two smaller circles of radii a and b are drawn, as shown, so that the smaller circles are tangent to the larger circle at P and Q. The smaller circles intersect at S and T. If P, S, and Q are collinear (that is, they lie on the same straight line), prove that $r = a + b$.



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M388. *Proposé par Kyle Sampson, St. John's, NL.*

On engendre une suite en écrivant (en ordre croissant), pour chaque entier n, les multiples de n jusqu'à et y compris n^2. La suite commence donc ainsi : $1, 2, 4, 3, 6, 9, 4, 8, 12, 16, 5, 10, 15, 20, 25, 6, 12, \ldots$. Déterminer le 2009^{e} terme de la suite.

M389. *Proposé par Lino Demasi, étudiant, Université Simon Fraser, Burnaby, BC.*

On a 2009 étudiants ayant chacun une carte portant un nombre entier positif différent. Si la somme des nombres figurant sur ces cartes est 2020049, quelles sont les valeurs possibles de la médiane des nombres sur ces cartes?

M390. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Un triangle pythagorique est un triangle rectangle dont les côtés et l'hypoténuse sont mesurés par des entiers. Soit a et b les longueurs des côtés d'un triangle pythagorique et h la longueur de la hauteur abaissée sur l'hypoténuse. Trouver tous les triangles de cette forme pour lesquels

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1 .$$

M391. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

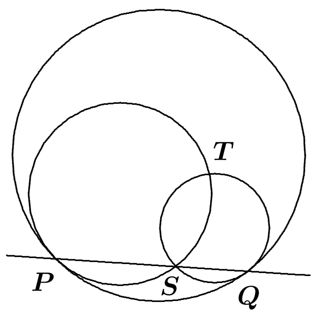
Trouver toutes les paires (a, b) de nombres entiers positifs pour lesquels $\frac{a+1}{b}$ et $\frac{b+2}{a}$ sont des entiers positifs.

M392. *Proposé par l'Équipe de Mayhem.*

Trouver, avec preuve à l'appui, la fraction $\frac{p}{q}$, avec p et q entiers positifs et $q < 1000$ et qui soit la plus proche de $\frac{19}{72}$, mais distincte de celle-ci.

M393. *Proposé par l'Équipe de Mayhem.*

Comme le montre la figure, on dessine deux petits cercles de rayon a et b dans un grand cercle de rayon r de sorte qu'ils soient tangents à celui-ci en P et Q. Les petits cercles se coupent en S et T. Si P, S et Q sont colinéaires (c-à-d s'ils sont situés sur une même droite), montrer que $r = a + b$.



Mayhem Solutions

M350. *Proposed by the Mayhem Staff.*

Dean rides his bicycle from Coe Hill to Apsley. By distance, one-third of the route is uphill, one-third of the route is downhill, and the rest of the route is on flat ground. Dean rides uphill at an average speed of 16 km/h and on flat ground at an average speed of 24 km/h. If his average speed over the whole trip is 24 km/h, then what is his average speed while riding downhill?

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

Suppose that it took x hours for Dean to travel from Coe Hill to Apsley. His average speed on this trip is 24 km/h, so he travelled a total distance of $24x$ km. Also, the total distance for each of the uphill, downhill, and level sections of the trip was $\frac{1}{3}(24x) = 8x$ km.

Dean rode at a speed of 16 km/h for a distance of $8x$ km, so his time riding uphill was $\frac{8x}{16} = \frac{x}{2}$ hours.

Dean rode at a speed of 24 km/h for a distance of $8x$ km, so his time riding on flat ground was $\frac{8x}{24} = \frac{x}{3}$ hours.

Adding the times so far, we obtain $\frac{x}{2}+\frac{x}{3}=\frac{5x}{6}$ hours. Since the total time for the trip was x hours, this means that Dean rode downhill for $x-\frac{5x}{6}=\frac{x}{6}$ hours.

This means that his downhill speed was $\frac{8x}{x/6}=48$ km/h.

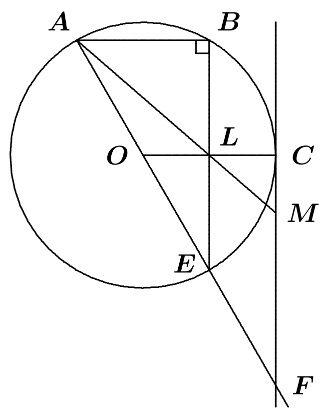
Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; ANTONIO GODOY TOHORIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; CARL LIBIS, University of Rhode Island, Kingston, RI, USA; KARAN PAHIL, student, Malcolm Munroe Junior High School, Sydney, NS; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and ALEX SONG, Elizabeth Ziegler Public School, Waterloo, ON.

M351. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

Let C be a point on a circle with centre O and radius r. The chord AB is of length r and is parallel to OC. The line AO cuts the circle again at E and it cuts the tangent to the circle at C at the point F. The chord BE cuts OC at L and AL cuts CF at M. Determine the ratio $CF : CM$.

Solution by Ian June L. Garces, Ateneo de Manila University, Quezon City, The Philippines.

Since AE is a diameter of the circle with centre O, we have that $\angle EBA = 90°$. Since AB is parallel to OC, we also have that $\angle BAE = \angle COF$. Since $\angle FCO$ is a right angle (because CF is the tangent to the circle at C) and $AB = OC = r$, then $\triangle ABE$ is congruent to $\triangle OCF$. Thus, $BE = CF$.



By the Pythagorean Theorem,

$$\begin{aligned} CF = BE &= \sqrt{AE^2 - AB^2} \\ &= \sqrt{(2r)^2 - r^2} = \sqrt{3}r\,. \end{aligned}$$

Again, since $\triangle ABE$ and $\triangle OCF$ are congruent, $AO+OE = AE = OF = OE+EF$, or $AO = OE = EF = r$, which means that E is the midpoint of OF.

Next, BE is parallel to CF because they are perpendicular to AB and OC, respectively, which are parallel. Since BE is parallel to CF and E is the midpoint of OF, it follows that L is also the midpoint of OC, whence $CL = \frac{1}{2}r$. Since BE is perpendicular to AB which is parallel to OC, then BE is perpendicular to OC as well.

Again since BE is parallel to CF, we deduce that $\angle CML = \angle BLA$. Also, $\angle MCL = \angle LBA = 90°$, and hence $\triangle MCL$ is similar to $\triangle LBA$.

Since $CL = \frac{1}{2}r = \frac{1}{2}AB$, we have that $CM = \frac{1}{2}BL$. Since also $BL = \frac{1}{2}BE$ (because a radius perpendicular to a chord bisects the chord), we have that $CM = \frac{1}{4}BE = \frac{\sqrt{3}}{4}r$.

Therefore, $\frac{CF}{CM} = \frac{\sqrt{3}r}{\sqrt{3}r/4} = 4$ and $CF : CM = 4 : 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; ANTONIO GODOY TOHORIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, Elizabeth Ziegler Public School, Waterloo, ON; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON. There was one incorrect and one incomplete solution submitted.

Garces and Hess also considered the case of BE intersecting OC extended, rather than OC itself.

M352. *Proposed by the Mayhem Staff.*

Consider the numbers 37, 44, 51, $\dots$, 177, which form an arithmetic sequence. A number n is the sum of five distinct numbers from this sequence. How many possible values of n are there?

Solution by Ian June L. Garces, Ateneo de Manila University, Quezon City, The Philippines.

The common difference of the given sequence is 7, and there are 21 terms in the sequence, since $\frac{177-37}{7} = 20$. Let n be a sum of five terms in the sequence; that is,

$$\begin{aligned} n &= (37+7a)+(37+7b)+(37+7c)+(37+7d)+(37+7e) \\ &= 185+7(a+b+c+d+e)\,, \end{aligned}$$

where a, b, c, d and e are distinct elements of the set $\{0, 1, 2, \dots, 19, 20\}$. Let $X = a+b+c+d+e$. Since we want to count the number of different possible values of n, it suffices to count the number of possible values of X.

The least possible value of X is $0+1+2+3+4 = 10$, and the largest possible value of X is $16+17+18+19+20 = 90$. We show that the integers from 10 to 90 inclusive are possible values of X.

When $a = 0$, $b = 1$, $c = 2$, $d = 3$, and $e = 4, 5, \dots, 19, 20$, the values of X range from 10 to 26.

When $a = 0$, $b = 1$, $c = 2$, $e = 20$, and $d = 4, 5, \dots, 18, 19$, the values of X range from 27 to 42.

When $a = 0$, $b = 1$, $d = 19$, $e = 20$, and $c = 3, 4, \dots, 17, 18$, the values of X range from 43 to 58.

When $a = 0$, $c = 18$, $d = 19$, $e = 20$, and $b = 2, 3, \dots, 16, 17$, the values of X range from 59 to 74.

When $b = 17$, $c = 18$, $d = 19$, $e = 20$, and $a = 1, 2, \dots, 15, 16$, the values of X range from 75 to 90.

Thus, every integer from 10 to 90 is a possible value of X. Since 10 and 90 are the smallest and largest possible values of X, respectively, there are $90 - 10 + 1 = 81$ possible values of X and so 81 possible values of n.

Also solved by LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA. There were four incorrect and four incomplete solutions submitted.

M353. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all pairs (x, y) of real numbers for which

$$xy + \frac{1}{x} + \frac{1}{y} = \frac{1}{xy} + x + y \,.$$

Solution by Antonio Godoy Tohoria, Madrid, Spain.

First, we note that $x \neq 0$ and $y \neq 0$. Next, we rearrange, multiply by xy and factor:

$$\begin{aligned}
xy + \frac{1}{x} + \frac{1}{y} - \frac{1}{xy} - x - y &= 0\,; \\
x^2y^2 + y + x - 1 - x^2y - xy^2 &= 0\,; \\
x^2y(y-1) + x\left(1 - y^2\right) + y - 1 &= 0\,; \\
(y-1)\left(x^2y - x(1+y) + 1\right) &= 0\,; \\
(y-1)\left(x^2y - x - xy + 1\right) &= 0\,; \\
(y-1)\big(xy(x-1) - (x-1)\big) &= 0\,; \\
(y-1)(x-1)(xy-1) &= 0\,.
\end{aligned}$$

Therefore, $x = 1$ or $y = 1$ or $xy = 1$.

Thus, the pairs which solve the equation are $(1, y)$ for any nonzero real number y, and $(x, 1)$ for any nonzero real number x, and $\left(t, \frac{1}{t}\right)$ for any nonzero real number t.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; CARL LIBIS, University of Rhode Island, Kingston, RI, USA; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, Elizabeth Ziegler Public School, Waterloo, ON; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON. There were two incorrect solutions submitted.

M354. *Proposed by the Mayhem Staff.*

Without using a calculating device, determine the prime factorization of $3^{20} + 3^{19} - 12$.

Solution by Luis De Sousa, student, IST-UTL, Lisbon, Portugal.

Factoring, we obtain

$$N = 3^{20}+3^{19}-12 = 3^{19}(3+1)-12 = 12\left(3^{18}-1\right) = 12\left(3^{9}+1\right)\left(3^{9}-1\right).$$

The identities $x^3-1=(x-1)(x^2+x+1)$ and $x^3+1=(x+1)(x^2-x+1)$, give us $3^9-1=(3^3-1)(3^6+3^3+1)$ and $3^9+1=(3^3+1)(3^6-3^3+1)$. Thus, $3^9-1=26\times 757$ and $3^9+1=28\times 703$.

So far, $N = 12\cdot 28\cdot 757\cdot 26\cdot 703 = 2^5\cdot 3\cdot 7\cdot 13\cdot 703\cdot 757$.

Lastly, we need to check if 703 and 757 are prime. Since $30^2=900$, we only need to check divisibility by primes less than 30. By brute force, we obtain that $703=19\cdot 37$ and that 757 is prime.

Therefore, the prime factorization is $N=2^5\cdot 3\cdot 7\cdot 13\cdot 19\cdot 37\cdot 757$.

Also solved by ROBERT BILINSKI, Collège Montmorency, Laval, QC; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; ANTONIO GODOY TOHORIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; R. LAUMEN, Deurne, Belgium; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There were two incorrect solutions submitted.

Some calculating can be saved by noting that $3^6-3^3+1 = 3^6+2\cdot 3^3+1-3\cdot 3^3 = \left(3^3+1\right)^2-3^4 = (3^3+1-3^2)(3^3+1+3^2) = 19\cdot 37$.

M355. *Proposed by the Mayhem Staff.*

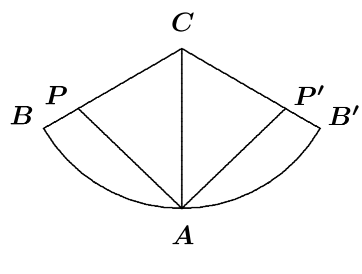
A right circular cone with vertex C has a base with radius 8 and a slant height of 24. Points A and B are diametrically opposite points on the circumference of the base. Point P lies on CB.

(a) If $CP=18$, determine the shortest path from A through P and back to A that travels completely around the cone.

(b) Determine the position of P on CB that minimizes the length of the shortest path in part (a).

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON.

Cut the cone along CB and flatten out the lateral surface. We label the endpoints of the arc which previously formed the circumference of the base as B and B'. Note that BCB' is a sector of the circle with radius $CB=CB'=24$. We also label the points on CB and CB' that are at a distance of 18 from C as P and P', respectively.



Since the cone had radius 8, then the length of arc BB' is $2\pi(8)=16\pi$. The entire circumference of the circle with centre C and radii BC and $B'C$ is $2\pi(24)=48\pi$, so BB' is one-third of the total circumference, and so

$\angle BCB' = 120^\circ$. Since A is the midpoint of arc BB', then $\angle BCA = 60^\circ$. The shortest path from A to P and back to A consists of the line segments AP and $P'A$, as the shortest distance between two points is a straight line.

By symmetry, $AP = P'A$. To find the length of AP, we use the Law of Cosines in $\triangle PCA$, whence

$$\begin{aligned} AP &= \sqrt{CP^2 + CA^2 - 2(CP)(CA)\cos(\angle PCA)} \\ &= \sqrt{18^2 + 24^2 - 2(18)(24)\cos(60^\circ)} \\ &= \sqrt{324 + 576 - 432} \;=\; \sqrt{468} \;=\; 6\sqrt{13}\,. \end{aligned}$$

Therefore, the length of the shortest path in part (a) is $2AP = 12\sqrt{13}$.

If $CP = x$ we again use the Law of Cosines to find the length of the shortest path from A to P and back to A again, which is $2AP$, so we compute

$$\begin{aligned} AP &= \sqrt{CP^2 + CA^2 - 2(CP)(CA)\cos(\angle PCA)} \\ &= \sqrt{x^2 + 24^2 - 2x(24)\cos(60^\circ)} \;=\; \sqrt{x^2 - 24x + 576}\,. \end{aligned}$$

Since this length equals $\sqrt{(x-12)^2 + 432}$, the length of the path is minimized when $x = 12$, so the length of the shortest path in part (b) is minimized when P is the midpoint of CB.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and RICARD PEIRÓ, IES "Abastos", Valencia, Spain. There was one incomplete solution submitted.

For P on the line BC, the length of AP is shortest when $AP \perp BC$. Since $\triangle ABC$ is equilateral, this occurs when P is the midpoint of BC, in agreement with the featured solution.

M356. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all pairs (k, n) of positive integers for which

$$k(k+1)(k+2)(k+3) \;=\; n(n+1)\,.$$

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON.

Note that

$$\begin{aligned} k(k+1)(k+2)(k+3) &= k(k+3)(k+1)(k+2) \\ &= \left(k^2 + 3k\right)\left(k^2 + 3k + 2\right) \\ &= \left[\left(k^2 + 3k + 1\right) - 1\right]\left[\left(k^2 + 3k + 1\right) + 1\right] \\ &= \left(k^2 + 3k + 1\right)^2 - 1 \;=\; m^2 - 1\,, \end{aligned}$$

where $m = k^2 + 3k + 1$. Since k is a positive integer, m is a positive integer.

Now $k(k+1)(k+2)(k+3) = n^2 + n$, so $m^2 - 1 = n^2 + n$ or $m^2 = n^2 + n + 1$. Since $n^2 < n^2 + n + 1 < n^2 + 2n + 1 = (n+1)^2$, it follows that $n^2 < m^2 < (n+1)^2$, which is not possible since n^2 and $(n+1)^2$ are consecutive squares.

Therefore, there are no solutions.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There were three incomplete solutions submitted.

Problem of the Month

Ian VanderBurgh

Have you ever looked at two things and tried to figure out which is taller? Maybe two friends or two trees? This isn't so hard if the two things are right next to each other, but to compare that maple tree in your backyard with the the oak tree in your front yard isn't that easy, since moving the trees is difficult, unless of course you're in a production of "the Scottish play". With two friends, you could get them to stand "back-to-back" and compare them, but what if the objects are not easily moveable? Some of us might be tempted to use trigonometry or some other advanced techniques.

There is another good way to do this – compare them to a "standard". This could be your house, a long stick that you have, or maybe another tree part way in between – basically, against anything that is easy to compare to each of them.

How does this relate to mathematics?

Problem 1 (2002 UK Intermediate Challenge). Given that $x = \frac{111110}{111111}$, $y = \frac{222221}{222223}$, $z = \frac{333331}{333334}$, which of the following statements is correct?

(A) $x < y < z$ (B) $x < z < y$ (C) $y < z < x$
(D) $z < x < y$ (E) $y < x < z$

Now wait – no calculators allowed! What could we do? We could try some long division. We could guess wildly. We could try comparing one of these fractions to another of these and do some arithmetical manipulations.

Or, we could compare them to a common standard. Can you see a "nice" number that is close to each of x, y, and z?

Solution. Each of x, y, and z is close to 1, so let's see how far each is from 1 and compare them this way:

$$x = 1 - \frac{1}{111111}\,; \qquad y = 1 - \frac{2}{222223}\,; \qquad z = 1 - \frac{3}{333334}\,.$$

So we've done an initial comparison of each of x, y, and z to 1. Can you tell which is the biggest now and which is the smallest?

Perhaps my brother (the smart one in the family!) could tell, but I'm not that quick. We've compared each to a common standard (that is, to 1) but now what about these differences? Again, there are lots of ways to do this, but let's try a variation on the common standard approach.

If we wrote each of these differences as a fraction with numerator 1, we could compare them relatively easily by comparing the sizes of the denominators. Let's try this. First, we rewrite these as

$$x = 1 - \frac{1}{111111}\,; \qquad y = 1 - \frac{1}{\left(\frac{222223}{2}\right)}\,; \qquad z = 1 - \frac{1}{\left(\frac{333334}{3}\right)}\,;$$

and then we convert each of the denominators to a mixed fraction:

$$x = 1 - \frac{1}{111111}; \qquad y = 1 - \frac{1}{111111\frac{1}{2}}; \qquad z = 1 - \frac{1}{111111\frac{1}{3}}.$$

Now, can you compare the denominators? With a little bit of thought, we can see that $111111 < 111111\frac{1}{3} < 111111\frac{1}{2}$.

This means that $\frac{1}{111111} > \frac{1}{111111\frac{1}{3}} > \frac{1}{111111\frac{1}{2}}$. So x is the furthest away from 1, since its difference with 1 is the largest. Similarly, y is the closest to 1, since its difference with 1 is the smallest. This tells us that $x < z < y$, so answer (B) is correct.

It's always satisfying to be able to answer this type of problem without using either a calculator or any algebra. Here's another problem that can use this "common standard" approach.

Problem 2 (1999 Pascal Contest). If $w = 2^{129} \cdot 3^{81} \cdot 5^{128}$, $x = 2^{127} \cdot 3^{81} \cdot 5^{128}$, $y = 2^{126} \cdot 3^{82} \cdot 5^{128}$, and $z = 2^{125} \cdot 3^{82} \cdot 5^{129}$, then the order from smallest to largest is

(A) w, x, y, z (B) x, w, y, z (C) x, y, z, w
(D) z, y, x, w (E) x, w, z, y

Here, your calculator wouldn't do you much good, as these numbers are likely way too big for your calculator to handle. So let's again try the "common standard" technique. But what is our common standard going to be?

Solution. We pick a common standard, N, to be the product of the smallest power of each of 2, 3, and 5 that occurs in each of the four original numbers. (Some of you may recognize N as the greatest common divisor of w, x, y, and z.) Among the four numbers, the smallest power of 2 that occurs is 2^{125}, the smallest power of 3 that occurs is 3^{81}, and the smallest power of 5 that occurs is 5^{128}. So we define $N = 2^{125} \cdot 3^{81} \cdot 5^{128}$.

How do we compare N to each of w, x, y and z? Should we use subtraction again? It actually makes more sense to use multiplication (or division, depending on your perspective).

Let's first compare N to w. Since N contains 125 factors of 2 and w contains 129 factors of 2, then we need to multiply N by 2^4 to get the correct number of factors of 2 for w. Since N contains 81 factors of 3 and w contains 81 factors of 3, then N already gives us the correct number of factors of 3 for w. Since N contains 128 factors of 5 and w contains 128 factors of 5, then N already gives us the correct numbers of factors of 5 for w. Thus, $w = (2^{125} \cdot 3^{81} \cdot 5^{128}) \cdot 2^4 = N \cdot 2^4$.

Similarly, $x = N \cdot 2^2$ and $y = N \cdot 2^1 \cdot 3^1$ and $z = N \cdot 3^1 \cdot 5^1$.

Put another way, $w = 16N$, $x = 4N$, $y = 6N$, and $z = 15N$. Since N is positive, $4N < 6N < 15N < 16N$, or $x < y < z < w$, so answer (C) is correct.

THE OLYMPIAD CORNER

No. 277

R.E. Woodrow

We begin with the Team Selection Examination for the International Mathematical Olympiad from the Scientific and Technical Research Institute of Turkey. My thanks for obtaining the problems for the *Corner* to Robert Morewood, Canadian Team Leader to the 47^{th} IMO in Slovenia.

The Scientific and Technical Research Institute of Turkey
Team Selection Examination for the International Mathematical Olympiad

First Day (1 April 2006)

1. Find the largest area of a heptagon two of whose diagonals are perpendicular and whose vertices lie on a unit circle.

2. Let n be a positive integer. In how many different ways can a $2 \times n$ rectangle be partitioned into rectangles with sides of integer length?

3. Let x, y, z be positive real numbers with $xy + yz + zx = 1$. Prove that

$$\frac{27}{4}(x+y)(y+z)(z+x) \;\geq\; \left(\sqrt{x+y}+\sqrt{y+x}+\sqrt{z+x}\,\right)^2 \;\geq\; 6\sqrt{3}\,.$$

Second Day (2 April 2006)

4. Find the smallest positive integer x_1 such that 2006 divides x_{2006}, if $x_{n+1} = x_1^2 + x_2^2 + \cdots + x_n^2$ for each integer $n \geq 1$.

5. Given a circle with diameter AB and a point Q on the circle different from A and B, let H be the foot of the perpendicular dropped from Q to AB. Prove that if the circle with centre Q and radius QH intersects the circle with diameter AB at C and D, then CD bisects QH.

6. In a university entrance examination with 2006000 students, each student makes a list of 12 colleges from a total of 2006 colleges. It turns out that for any 6 students, there exist two colleges such that each of the 6 students included at least one of these two colleges on his or her list. An *extensive list* is a list which includes at least one college from each student's list.

(a) Prove that there exists an extensive list of 12 colleges.

(b) Prove that the students can choose their lists so that no extensive list of fewer than 12 colleges can be found.

Next we give the two days of the XIII National Mathematical Olympiad of Turkey, given under the auspices of the Scientific and Technical Research Institute of Turkey.

The Scientific and Technical Research Institute of Turkey
XIII National Mathematical Olympiad
Second Round

First Day (10 December 2005)

1. Let a, b, c, and d be real numbers. Prove that

$$\sqrt{a^4+c^4}+\sqrt{a^4+d^4}+\sqrt{b^4+c^4}+\sqrt{b^4+d^4} \;\geq\; 2\sqrt{2}(ab+bc)\,.$$

2. In a triangle ABC with $|AB|<|AC|<|BC|$, the perpendicular bisector of AC intersects BC at K and the perpendicular bisector of BC intersects AC at L. Let O, O_1, and O_2 be the circumcentres of the triangles ABC, CKL, and OAB, respectively. Prove that OCO_1O_2 is a parallelogram.

3. Some of the $n+1$ cities in a country (including the capital city) are connected by one-way or two-way airlines. No two cities are connected by both a one-way airline and a two-way airline, but there may be more than one two-way airline between two cities. If d_A denotes the number of airlines flying from a city A, then $d_A \leq n$ for any city A other than the capital city and $d_A + d_B \leq n$ for any two cities A and B other than the capital city and which are not connected by a two-way airline. Every airline has a return, possibly consisting of several connecting flights. Find the largest possible number of two-way airlines and all configurations of airlines for which this largest number is attained.

Second Day (11 December 2005)

4. Find all triples (m,n,k) of nonnegative integers such that $5^m+7^n=k^3$.

5. Let a, b, and c be the side lengths of a triangle whose incircle has radius r. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \;\leq\; \frac{1}{4r^2}\,.$$

6. Let $\{a_n\}$ be a sequence of integers for which there exists a positive integer N such that for any $n \geq N$, $a_n = |\{i : 1 \leq i < n \text{ and } a_i + i \geq n\}|$. Determine the maximum number of distinct values that can be attained infinitely many times by this sequence.

Next we give the 2005 Australian Mathematical Olympiad, written in February 2005. Thanks for collecting these problems again goes to Robert Morewood, Canadian Team Leader to the 47^{th} IMO.

2005 Australian Mathematical Olympiad

Paper 1

1. Let ABC be a right-angled triangle with the right angle at C. Let $BCDE$ and $ACFG$ be squares external to the triangle. Furthermore, let AE intersect BC at H, and let BG intersect AC at K. Find the size of $\angle DKH$.

2. Consider a polyhedron whose faces are convex polygons. Show that it has at least two faces with the same number of edges.

3. Let n be a positive integer, and let a_1, a_2, $\dots$, a_n be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\frac{a_1}{a_1^2+1} + \frac{a_2}{a_2^2+1} + \cdots + \frac{a_n}{a_n^2+1} \le \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1}\,.$$

4. Prove that for each positive integer n there exists a positive integer x such that $\sqrt{x+2004^n} + \sqrt{x} = \left(\sqrt{2005}+1\right)^n$.

Paper 2

5. In a multiple choice test there are $q \ge 10$ questions. For each question there are $a > 1$ possible answers, exactly one of which is right. A student who gets r answers right, w answers wrong and does not attempt the other questions will receive a score of $\dfrac{100(r-w)}{q(a-1)}$. Determine the pairs (q, a) for which all possible scores are integers.

6. Let ABC be a triangle. Let D, E, and F be points on the line segments BC, CA, and AB, respectively, such that line segments AD, BE, and CF meet in a single point. Suppose that $ACDF$ and $BCEF$ are cyclic quadrilaterals. Prove that AD is perpendicular to BC, BE is perpendicular to AC, and CF is perpendicular to AB.

7. Let a_0, a_1, a_2, $\dots$ and b_0, b_1, b_2, $\dots$ be two sequences of integers such that $a_0 = b_0 = 1$ and for each nonnegative integer k

(a) $a_{k+1} = b_0 + b_1 + b_2 + \cdots + b_k$, and

(b) $b_{k+1} = (0^2+0+1)a_0 + \left(1^2+1+1\right)a_1 + \cdots + \left(k^2+k+1\right)a_k$.

For each positive integer n show that

$$a_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n}\,.$$

8. In an $n \times n$ array, each of n distinct symbols occurs exactly n times. An example with $n = 3$ is shown at right.

Show that there is a row or a column in the array containing at least $\sqrt{n}$ distinct symbols.

1	2	3
1	3	2
3	2	1

Our next set of problems is the 56th Belarusian Mathematical Olympiad, Category C, Final Round. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO, for collecting them for our use.

56th Belarusian Mathematical Olympiad 2006
Category C, Final Round

1. (E. Barabanov) Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets

(a) there is a number c in the third subset such that $a + b = 2c$?

(b) there are numbers c_1 and c_2 in the third subset such that $a+b = c_1+c_2$?

2. (S. Mazanik) The points X, Y, and Z lie on the sides AB, BC, and CD of the rhombus $ABCD$, respectively, so that $XY||AZ$. Prove that XZ, AY, and BD are concurrent.

3. (V. Karamzin) Let a, b, and c be positive real numbers such that $abc = 1$. Prove that $2(a^2 + b^2 + c^2) + a + b + c \geq ab + bc + ca + 6$.

4. (D. Dudko) Triangle ABC has $\angle A = 60°$, $AB = 2005$, and $AC = 2006$. Alice and Betty take turns cutting the triangle with Alice going first. A player may cut a triangle along any straight line provided that two new triangles are formed and each has area at least 1. After each move an obtuse-angled triangle (or any one of two right-angled triangles) is removed and the next player cuts the remaining triangle. A player loses if she cannot move. Which player has a winning strategy?

5. (I. Voronovich) Let AA_1, BB_1, and CC_1 be the altitudes of an acute triangle ABC. Prove that the feet of the perpendiculars from C_1 to the segments AC, BC, BB_1, and AA_1 are collinear.

6. (V. Karamzin) Let a, b, and k be real numbers with $k > 0$. A circle with centre (a, b) has at least three common points with the parabola $y = k^2$: one of them is the origin $(0, 0)$ and two of the others lie on the line $y = kx + b$. Prove that $b \geq 2$.

7. (I. Zhuk) Let x, y, and z be real numbers greater than 1 such that

$$\begin{aligned} xy^2 - y^2 + 4xy + 4x - 4y &= 4004\,, \\ xz^2 - z^2 + 6xz + 9x - 6z &= 1009\,. \end{aligned}$$

Determine all possible values of $xyz + 3xy + 2xz - yz + 6x - 3y - 2z$.

8. (I. Akulich) A $2n \times 2n$ square is divided into $4n^2$ unit squares. What is the greatest possible number of diagonals of these unit squares one can draw so that no two of them have a point in common (including the endpoints of the diagonals)?

As a final set for this number we give Category B of the 56th Belarusian Mathematical Olympiad 2006. Again thanks go to Robert Morewood for obtaining them for our use.

56th Belarusian Mathematical Olympiad 2006
Category B, Final Round

1. (I.Voronovich) Given a convex quadrilateral $ABCD$ with $DC = a$, $BC = b$, $\angle DAB = 90^\circ$, $\angle DCB = \varphi$, and $AB = AD$, find the length of the diagonal AC.

2. (E.Barabanov) Is it possible to partition the set S into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets the number $2(a + b)$ belongs to the third subset, if

(a) S is the set of all integers?

(b) S is the set of all rational numbers?

3. (I. Biznets) Let a, b, and c be positive real numbers. Prove that

$$\frac{a^3 - 2a + 2}{b + c} + \frac{b^3 - 2b + 2}{c + a} + \frac{c^3 - 2c + 2}{a + b} \geq \frac{3}{2}\,.$$

4. (I. Voronovich) Let a and b be positive integers such that $a + 77b$ is divisible by 79 and $a + 79b$ is divisible by 77. Find the smallest possible value of the sum $a + b$.

5. (I. Zhuk) Three distinct points A, B, and C lie on the parabola $y = x^2$. Let R be the circumradius of the triangle ABC.

(a) Prove that $R \geq \frac{1}{2}$.

(b) Does there exist a constant $c > \frac{1}{2}$ such that for any three distinct points A, B, and C on the parabola $y = x^2$ the inequality $R \geq c$ holds?

6. (I. Voronovich) A sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ of pairs of real numbers is such that $(a_{n+1}, b_{n+1}) = \left(a_n^2 - 2b_n, b_n^2 - 2a_n\right)$ for all $n \geq 1$. Find $2^{512}a_{10} - b_{10}$ if $4a_1 - 2b_1 = 7$.

7. (I. Voronovich) The point K (distinct from the orthocentre) lies on the altitude CC_1 of the acute triangle ABC. Prove that the feet of the perpendiculars from C_1 to the segments AC, BC, BK, and AK lie on a circle.

8. (E. Barabanov, V. Kaskevich, S. Mazanik, I. Voronovich) An equilateral triangle of side n is divided into n^2 unit equilateral triangles by lines parallel to its sides. Determine the smallest possible number of small triangles that must be marked so that any unmarked triangle has at least one side in common with a marked triangle.

Next we turn to the file of solutions from our readers to problems given in the May 2008 number of the *Corner*, starting with the Mathematical Competition Baltic Way 2004 given at [2008 : 211–213].

1. Let a_1, a_2, a_3, ... be a sequence of nonnegative real numbers such that for each $n \geq 1$ both $a_n + a_{2n} \geq 3n$ and $a_{n+1} + n \leq 2\sqrt{(n+1)a_n}$ hold.

(a) Prove that $a_n \geq n$ for each $n \geq 1$.

(b) Give an example of such a sequence.

Solved by Jean-David Houle, student, McGill University, Montreal, QC; Pavlos Maragoudakis, Pireas, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write up.

(a) Suppose that $a_n < n$ for some n. Then

$$\begin{aligned} a_{n+1} + n &\leq 2\sqrt{(n+1)a_n} \;<\; 2\sqrt{(n+1)n} \\ &= \sqrt{4n^2+4n} \;<\; \sqrt{4n^2+4n+1} \;=\; 2n+1 \end{aligned}$$

implies that $a_{n+1} < n + 1$.

Now suppose that $a_{n+k} < n + k$ for some $k \geq 1$. Then

$$\begin{aligned} a_{n+k+1} + (n+k) &\leq 2\sqrt{(n+k+1)a_{n+k}} \;<\; 2\sqrt{(n+k+1)(n+k)} \\ &= \sqrt{4(n+k)^2 + 4(n+k)} \\ &< \sqrt{4(n+k)^2 + 4(n+k) + 1} \;=\; 2n + 2k + 1 \end{aligned}$$

implies that $a_{n+k+1} < n + k + 1$. By induction on k, we conclude that $a_{n+k} < n+k$ for each integer $k \geq 1$. In particular, $a_n + a_{2n} < n + 2n = 3n$, a contradiction. Therefore, $a_n \geq n$ for all n.

(b) If $a_n = n+1$ for each n, then $a_n + a_{2n} = 3n+3 > 3n$ and also $a_{n+1} + n = 2n+2 = 2\sqrt{(n+1)a_n}$. Hence, $\{a_n\}_1^\infty = \{n+1\}_1^\infty$ is an example of such a sequence.

2. Let $P(x)$ be a polynomial with nonnegative coefficients. Prove that if

$$P\left(\frac{1}{x}\right)P(x) \ \geq\ 1$$

for $x = 1$, then the same inequality holds for each positive x.

Solved by George Apostolopoulos, Messolonghi, Greece; Jean-David Houle, student, McGill University, Montreal, QC; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Apostolopoulos.

Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$. For $x = 1$ we have $P(1)P(1) \geq 1$, so $P(1)^2 = (a_n + a_{n-1} + \cdots + a_0)^2 \geq 1$.

For $x > 0$ let

$$\begin{aligned} \vec{u} &= \left(\sqrt{a_nx^n}, \sqrt{a_{n-1}x^{n-1}}, \ldots, \sqrt{a_0}\right), \\ \vec{v} &= \left(\sqrt{\frac{a_n}{x^n}}, \sqrt{\frac{a_{n-1}}{x^{n-1}}}, \ldots, \sqrt{a_0}\right). \end{aligned}$$

Applying the Cauchy–Schwarz Inequality to $\vec{u}$ and $\vec{v}$ yields

$$\begin{aligned} &P(x)P(\tfrac{1}{x}) \\ &= \ (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0)\left(\frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \cdots + a_0\right) \\ &\geq\ (a_n + a_{n-1} + \cdots + a_1 + a_0)^2 \ \geq\ 1\,. \end{aligned}$$

3. Let p, q, and r be positive real numbers and let n be a positive integer. If $pqr = 1$, prove that

$$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \ \leq\ 1\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's version.

Let $p^n = x^3$, $q^n = y^3$, and $r^n = z^3$. We must prove that if $xyz = 1$, then

$$\frac{1}{x^3 + y^3 + 1} + \frac{1}{y^3 + z^3 + 1} + \frac{1}{z^3 + x^3 + 1} \ \leq\ 1\,,$$

or equivalently we must prove that

$$\frac{1}{x^3+y^3+xyz} + \frac{1}{y^3+z^3+xyz} + \frac{1}{z^3+y^3+xyz} \le 1\,.$$

Since x and y are positive, $(x-y)^2(x+y) \ge 0$, hence $(x-y)(x^2-y^2) \ge 0$, hence $x^3+y^3 \ge xy(x+y)$. It follows that

$$\begin{aligned} & \frac{1}{x^3+y^3+xyz} + \frac{1}{y^3+z^3+xyz} + \frac{1}{z^3+x^3+xyz} \\ \le\ & \frac{1}{xy(x+y)+xyz} + \frac{1}{yz(y+z)+xyz} + \frac{1}{zx(z+x)+xyz} \\ =\ & \frac{x+y+z}{xyz(x+y+z)} = 1\,. \end{aligned}$$

Equality holds if and only if $x=y=z=1$, or if and only if $p=q=r=1$.

4. Let $\{x_1, x_2, \dots, x_n\}$ be a set of real numbers with arithmetic mean X. Prove that there is a positive integer K such that the arithmetic mean of each of the sets

$$\{x_1,x_2,\dots,x_K\}\,,\ \{x_2,x_3,\dots,x_K\}\,,\ \dots\,,\ \{x_{K-1},x_K\}\,,\ \{x_K\}$$

is not greater than X.

Solved by Oliver Geupel, Brühl, NRW, Germany; Jean-David Houle, student, McGill University, Montreal, QC; and Titu Zvonaru, Comănești, Romania. We give Houle's approach.

Suppose otherwise. That is, for each integer k with $1 \le k \le n$ suppose that one of the sets [*Ed.: indeed the problem statement and the following argument can be adapted for* sequences $x_1, x_2, \dots, x_n$.]

$$\{x_1,x_2,\dots,x_k\}\,,\ \{x_2,x_3,\dots,x_k\}\,,\ \dots\,,\ \{x_{k-1},x_k\}\,,\ \{x_k\}$$

has arithmetic mean greater than X. Now we show by induction on k that this implies that the mean of $\{x_1, x_2, \dots, x_k\}$ is greater than X for each k, which is a contradiction.

For $k=1$ it is obvious that $x_1 > X$ by our assumption. Assume that the result holds for all positive integers less than some $k>1$. Then there is an $i \le k$ such that the set $\{x_i, x_{i+1}, \dots, x_k\}$ has a mean greater than X. But we know that $\{x_1, x_2, \dots, x_{i-1}\}$ is either empty or has a mean greater than X, hence

$$\begin{aligned} x_1+x_2+\cdots+x_k &= (x_1+\cdots+x_{i-1}) + (x_i+\cdots+x_k) \\ &> (i-1)X + (k-i+1)X = kX\,, \end{aligned}$$

so that $\dfrac{x_1+\cdots+x_k}{k} > X$, as required.

5. For integers k and n let $(k)_{2n+1}$ be the multiple of $2n+1$ closest to k. Determine the range of the function $f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k$.

Solution by Titu Zvonaru, Comănești, Romania.

We have $|(k)_{2n+1} - k| \leq n$, hence

$$\begin{aligned} |f(k)| &= |(k)_3 - k + (2k)_5 - 2k + (3k)_7 - 3k| \\ &\leq |(k)_3 - k| + |(2k)_5 - 2k| + |(3k)_7 - 3k| \ \leq\ 1+2+3 \ =\ 6\,. \end{aligned}$$

It follows that $f(\mathbb{Z}) \subset \{0,\ \pm 1,\ \pm 2,\ \dots,\ \pm 6\}$. Note also that f is an odd function, that is, $f(-k) = -f(k)$ for each k. By direct computation we find that $f(0) = 0$, $f(6) = 1$, $f(16) = -2$, $f(3) = -3$, $f(20) = 4$, $f(31) = -5$, and $f(1) = -6$. It follows that the range of f is $\{0,\ \pm 1,\ \pm 2,\ \dots,\ \pm 6\}$.

6. A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three adjacent faces. The sum of these products is 1001. What is the sum of the six numbers on the faces?

Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; Jean-David Houle, student, McGill University, Montreal, QC; and Titu Zvonaru, Comănești, Romania. We give the write up of Houle.

Let the numbers on the faces be x_1, x_2, $\dots$, x_6 such that x_1 and x_6 are on opposite faces, as are x_2 and x_5, and x_3 and x_4. Then we have

$$\begin{aligned} x_1x_2x_3 + x_1x_3x_5 + x_1x_5x_4 + x_1x_4x_2 & \\ + x_6x_2x_3 + x_6x_3x_5 + x_6x_5x_4 + x_6x_4x_2 &= 1001\,; \\ (x_1+x_6)(x_2x_3 + x_3x_5 + x_5x_4 + x_4x_2) &= 1001\,; \\ (x_1+x_6)(x_3+x_4)(x_2+x_5) &= 1001 \ =\ 7 \cdot 11 \cdot 13\,. \end{aligned}$$

Since 7, 11, and 13 are primes and (x_1+x_6), (x_3+x_4), and (x_2+x_5) are integers greater than 1, the latter are the former in some order. it follows that the desired sum is $7+11+13 = 31$.

7. Find all sets X consisting of at least two positive integers such that for every pair $m{,}n \in X$, where $n > m$, there exists $k \in X$ such that $n = mk^2$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution by Apostolopoulos.

Let the set X satisfy the conditions and let m and n, where $m < n$, be the two smallest elements in the set X. There is a $k \in X$ so that $n = mk^2$, but as $m \leq k \leq n$, either $k = n$ or $k = m$. In the first case $mn = 1$, a contradiction. In the second case $n = m^3$ with $m > 1$. Suppose $|X| \geq 3$ and let $q \in X$ be the third smallest element. Then there is $k_0 \in X$ such that $q = mk_0^2$. We have $q > k_0$, so $k_0 = m$ or $k_0 = n$. However, $k_0 = m$

implies $q = n$, a contradiction, thus $k_0 = n = m^3$ and $q = m^7$. Now there exists $k_1 \in X$ such that $q = nk_1^2$, hence $k_1 = m^2$. Since $m^2 \notin X$, we have a contradiction if $|X| \geq 3$.

Therefore, the only elements that the set X can contain are m and m^3 for some $m > 1$.

8. Let $f(x)$ be a nonconstant polynomial with integer coefficients. Prove that there is an integer n such that $f(n)$ has at least 2004 distinct prime factors.

Solved by George Apostolopoulos, Messolonghi, Greece; and Jean-David Houle, student, McGill University, Montreal, QC. We give the argument of Apostolopoulos.

Suppose the contrary. Choose an integer n_0 such that $f(n_0)$ has the highest number of distinct prime factors. By translating the argument of the polynomial, we may assume that $n_0 = 0$. Setting $k = f(0)$, we have $f(wk^2) \equiv k \pmod{k^2}$, or $f(wk^2) = \alpha k^2 + k = (\alpha k + 1)k$. Now, since $\gcd(\alpha k + 1, k) = 1$ and k has the highest number of distinct prime factors of any admissible value of f, we must have $\alpha k + 1 = \pm 1$. This cannot happen for each w since f is nonconstant (in particular $|f(x)| \to \infty$ as $x \to \infty$) so our supposition leads to a contradiction.

10. Is there an infinite sequence of prime numbers $p_1, p_2, p_3, \ldots$ such that $|p_{n+1} - 2p_n| = 1$ for each $n \geq 1$?

Solution by George Apostolopoulos, Messolonghi, Greece.

No, there is no such sequence.

Suppose the contrary. Clearly $p_3 > 3$ and $p_3 \equiv \pm 1 \pmod 3$.

Further suppose that $p_3 \equiv 1 \pmod 3$. Then $p_4 = 2p_3 - 1$ (otherwise $p_4 \equiv 0 \pmod 3$), so $p_4 \equiv 1 \pmod 3$. Similarly, $p_5 = 2p_4 - 1$, $p_6 = 2p_5 - 1$, and so forth. By induction we have $p_{n+1} = 1 + 2^{n-2}(p_3 - 1)$ for $n \geq 3$. If we set $n = p_3 + 1$, then using Fermat's little theorem we have

$$p_{p_3+2} \;=\; 1 + 2^{p_3-1}(p_3 - 1) \;\equiv\; 1 + 1 \cdot (p_3 - 1) \;=\; p_3 \;\equiv\; 0 \pmod{p_3}\,,$$

a contradiction.

If $p_3 \equiv -1 \pmod 3$, then $p_{n+1} = -1 + 2^{n-2}(p_3 + 1)$ for $n \geq 3$ is obtained by similar arguments. Taking $n = p_3 + 1$ then leads again to the contradiction $p_{p_3+2} \equiv 0 \pmod{p_3}$.

11. An $m \times n$ table is given with $+1$ or -1 written in each cell. Initially there is exactly one -1 in the table and all the other cells contain a $+1$. A move consists of choosing a cell containing -1, replacing this -1 by a 0, and then multiplying all the numbers in the neighbouring cells by -1 (two cells are neighbouring if they share a side). For which (m, n) can a sequence of such moves always reduce the table to all zeros, regardless of which cell contains the initial -1?

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let (m,n) have the desired property. Perform a sequence of moves that reduces the table to all zeros, and let e_k be the total number of edges after the k^{th} move which border two zero cells. We will prove by induction that $e_k \equiv k-1 \pmod 2$ for $1 \le k \le mn$.

For $k=1$ this is clear. For the induction step, it suffices to prove that $e_{k+1} \equiv e_k - 1 \pmod 2$. Now, each time a cell changes from ± 1 to ∓ 1, one of its neighbours changes to a zero. Hence, if a cell initially contained $+1$, then whenever it contains -1 an odd number of its neighbours contain zeros. Therefore, if a -1 is changed to a zero, then an odd number of edges bordered by zeros are created. This means that e_{k+1} and e_k differ by an odd number, verifying the induction step.

We have

$$e_{mn} \;=\; (m-1)n + m(n-1) \;=\; 2mn - (m+n)$$

and by setting $k = mn$ in the relation $e_k \equiv k-1 \pmod 2$ we obtain

$$\begin{aligned} 2mn - (m+n) &\equiv mn - 1 \pmod 2 \,, \\ mn - m - n + 1 &\equiv 0 \pmod 2 \,, \\ (m-1)(n-1) &\equiv 0 \pmod 2 \,. \end{aligned}$$

Thus, m or n is odd.

Conversely, if m or n is odd, then we show how to reduce the table to all zeros. Let cell (i,j) be the cell in the i^{th} row and j^{th} column. Let n be odd and let the cell (k,ℓ) contain the initial -1. We reduce each of the cells (k,ℓ), $(k-1,\ell)$, $\ldots$, $(1,\ell)$ and also the cells $(k+1,\ell)$, $(k+2,\ell)$, $\ldots$, (m,ℓ) to zero, in the order given. Now the ℓ^{th} column is all zeros, whereas the $(\ell-1)^{\text{st}}$ and the $(\ell+1)^{\text{st}}$ columns (if present) are all -1's. A column with all -1 entries and an odd number n of cells is reduced to zero by changing the 1^{st}, 3^{rd}, 5^{th}, $\ldots$, n^{th} cells to zero followed by the 2^{nd}, 4^{th}, $\ldots$, $(n-1)^{\text{th}}$ cells to zero, leaving behind columns of -1's. In this way we can reduce the whole table to zeros.

13. The 25 member states of the European Union set up a committee with the following rules.

(a) The committee shall meet every day.

(b) At each meeting, at least one member state shall be represented.

(c) At any two different meetings, a different set of member states shall be represented.

(d) The set of states represented at the n^{th} meeting shall include, for every $k < n$, at least one state that was represented at the k^{th} meeting.

For how many days can the committee have its meetings?

Solution by George Apostolopoulos, Messolonghi, Greece.

If one member state is always represented, then rules (b) and (d) will be satisfied. There are 2^{24} different subsets of the remaining 24 member states, so there can be at least 2^{24} meetings. However, rule (c) forbids complimentary sets in two different meetings, so the number of meetings cannot exceed $\frac{1}{2} \cdot 2^{25} = 2^{24}$. Therefore, the committee can have its meetings for 2^{24} days.

14. A pile of one, two, or three nuts is *small*, while a pile of four or more nuts is *large*. Two persons play a game, starting with a pile of n nuts. A player moves by taking a large pile of nuts and splitting it into two non-empty piles (either pile can be large or small). If a player cannot move, he loses. For which values of n does the first player have a winning strategy?

Solution by Jean-David Houle, student, McGill University, Montreal, QC.

We prove by induction that the Sprague–Grundy function satisfies $g(n) \equiv n - 3 \pmod 4$, which shows that the second player has a winning strategy if and only if $n \equiv 3 \pmod 4$ or $n \leq 3$. It is straightforward to show that $g(0) = g(1) = g(2) = g(3) = 0$, $g(4) = 1$, $g(5) = 2$, $g(6) = 3$, and $g(7) = 0$.

Now suppose that $g(k) \equiv k - 3 \pmod 4$ for $k < n$.

Case 1 $n \equiv 0 \pmod 4$. By the induction hypothesis, we have that $g(x-1, 1) = g(x-1) \oplus g(1) = 0 \oplus 0 = 0$, since $x - 1 \equiv 3 \pmod 4$. Suppose there are a, b such that $a + b = n$ and $g(a, b) = g(a) \oplus g(b) = 1$. Then it follows that $|g(a) - g(b)| = 1$, hence $a + b \equiv 1 \pmod 2$, a contradiction. Thus, $g(n) = 1$.

Case 2 $n \equiv 1 \pmod 4$. By the induction hypothesis, we have $g(x-1, 1) = 1$ and $g(x-2, 2) = 0$, similarly as above. Suppose there are a, b such that $a + b = n$ and $g(a, b) = g(a) \oplus g(b) = 2$. It follows that $|g(a) - g(b)| = 2$, hence $a + b \equiv 0 \pmod 2$, a contradiction. Thus, $g(n) = 2$.

Case 3 $n \equiv 2 \pmod 4$. From the induction hypothesis, we have that $g(x-1, 1) = 2$, $g(x-2, 2) = 1$ and $g(x-3, 3) = 0$. Suppose there are a, b such that $a + b = n$ and $g(a, b) = g(a) \oplus g(b) = 3$. Then by induction we must have either $g(a) = 3$ and $g(b) = 0$, or $g(a) = 2$ and $g(b) = 1$, assuming for definiteness that $g(a) > g(b)$. But then $n \equiv 2 + 3 \pmod 4$ or $n \equiv 1 + 0 \pmod 4$, both contradictions. Thus, $g(n) = 3$.

Case 4 $n \equiv 3 \pmod 4$. If there are a, b such that $a + b = n$ and also $g(a, b) = g(a) \oplus g(b) = 0$, then it would follow that $g(a) = g(b)$. However, since $n \equiv 3 \pmod 4$, this is possible only if one of a, b is 1, 2, or 3. But then it would be impossible for the other to be 3 modulo 4. Thus, $g(n) = 0$.

[*Ed.: See* `http://www.math.ucla.edu/~tom/Game_Theory/comb.pdf` *for a gentle introduction to the* Sprague–Grundy function, *as well as the book* Winning Ways for your Mathematical Plays *(2nd ed., A K Peters, Ltd., 2001) by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy.*]

15. A circle is divided into 13 segments, numbered consecutively from 1 to 13. Five fleas called A, B, C, D, and E sit in the segments 1, 2, 3, 4, and 5, respectively. A flea can jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at a time, and two fleas cannot occupy the same segment. After some jumps, the fleas are back in the segments 1, 2, 3, 4, and 5, but possibly in some other order than they started in. Which orders are possible?

Solved by Oliver Geupel, Brühl, NRW, Germany; and Jean-David Houle, student, McGill University, Montreal, QC. We give Geupel's version.

The segments are labelled consecutively S_1, S_2, $\dots$, S_{13}. We define an alternative labelling T_1, T_2, $\dots$, T_{13} by the correspondence

$$\begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} \\ S_1 & S_6 & S_{11} & S_3 & S_8 & S_{13} & S_5 & S_{10} & S_2 & S_7 & S_{12} & S_4 & S_9 \end{pmatrix} .$$

The T-labelling is such that if a flea jumps five positions away, then its T-index changes by ± 1. Two fleas cannot occupy the same segment, hence the order of the fleas with respect to the T-numbering is invariant up to cyclic shifts. Initially, $\begin{pmatrix} S_1 & S_2 & S_3 & S_4 & S_5 \\ A & B & C & D & E \end{pmatrix}$ or $\begin{pmatrix} T_1 & T_4 & T_7 & T_9 & T_{12} \\ A & C & E & B & D \end{pmatrix}$ is the order of the fleas. Therefore, the possible finishing positions are

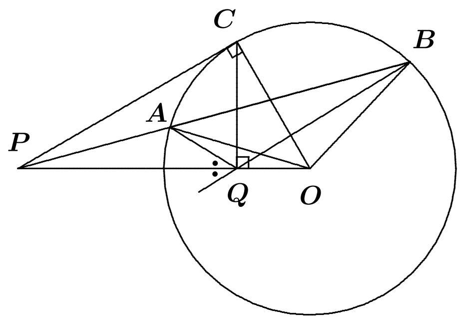
$$\begin{pmatrix} T_1 & T_4 & T_7 & T_9 & T_{12} \\ A & C & E & B & D \\ D & A & C & E & B \\ B & D & A & C & E \\ E & B & D & A & C \\ C & E & B & D & A \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} S_1 & S_2 & S_3 & S_4 & S_5 \\ A & B & C & D & E \\ D & E & A & B & C \\ B & C & D & E & A \\ E & A & B & C & D \\ C & D & E & A & B \end{pmatrix} ,$$

which are the five cyclic shifts of the initial ordering of the fleas.

We prove that these five orders are reachable. Let A, B, C, D, and E jump, once each, in the same direction. The new (consecutive) positions then have indices that are 5 more modulo 13 than the original indices, reduced to the range $1, 2, \dots, 13$. Since 5 and 13 are coprime, they eventually reach the position $\begin{pmatrix} S_2 & S_3 & S_4 & S_5 & S_6 \\ A & B & C & D & E \end{pmatrix}$. If E jumps backwards, then we obtain a cyclic shift of the starting order. The other required finishing positions are reached by iterating. The following generalization is now obvious: For s segments and n fleas ($n < s$) where a jump is n positions away, if s and n are coprime, then the possible finishing positions are the n cyclic shifts of the starting order.

16. Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of C onto the diameter is Q. Prove that QC bisects $\angle AQB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Babis Stergiou, Chalkida, Greece; and Titu Zvonaru, Comǎneşti, Romania. We give the solution by Amengual Covas.



Let O be the centre of the given circle.

By the theorem of the power of a point with respect to a circle applied to P,

$$PA \cdot PB = PC^2 .$$

Now, in right triangle CPO, CQ is the altitude to the hypotenuse. By a standard mean proportion we then have

$$PC^2 = PO \cdot PQ .$$

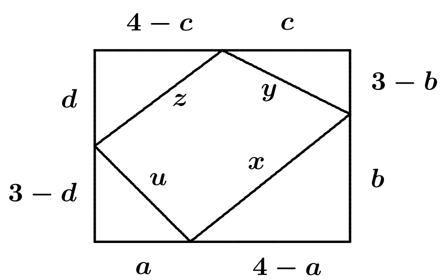
Hence,

$$PA \cdot PB = PO \cdot PQ ,$$

or equivalently, $\dfrac{PQ}{PA} = \dfrac{PB}{PO}$ and $\dfrac{PQ}{PB} = \dfrac{PA}{PO}$, so that triangles APQ, APO are similar to triangles BPQ, APO; respectively. Observing proportional sides, we have $\dfrac{AQ}{PA} = \dfrac{OB}{PO}$ and $\dfrac{QB}{PB} = \dfrac{OA}{PO}$. Since $OA = OB$, it follows that $\dfrac{AQ}{PA} = \dfrac{QB}{PB}$, hence PO is the external bisector of the angle at Q in triangle AQB. However $CQ \perp PQ$, so CQ is the internal bisector of the angle at Q in triangle AQB, yielding $\angle AQC = \angle CQB$, as required.

17. Consider a rectangle with sides of lengths 3 and 4, and on each side pick an arbitrary point that is not a corner. Let x, y, z, and u be the lengths of the sides of the quadrilateral spanned by these points. Prove that

$$25 \le x^2 + y^2 + z^2 + u^2 \le 50 .$$



Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; Jean-David Houle, student, McGill University, Montreal, QC; and Titu Zvonaru, Comǎneşti, Romania. We give the presentation of Geupel.

We assume that the four points divide the respective sides in the ratios $\dfrac{a}{4-a}$, $\dfrac{b}{3-b}$, $\dfrac{c}{4-c}$, and $\dfrac{d}{3-d}$.

We have

$$\begin{aligned} & x^2+y^2+z^2+u^2 \\ & \quad = \quad (4-a)^2+b^2+(3-b)^2+c^2+(4-c)^2+d^2+(3-d)^2+a^2 \\ & \quad = \quad 25+2(a-2)^2+2\left(b-\frac{3}{2}\right)^2+2(c-2)^2+2\left(d-\frac{3}{2}\right)^2 . \end{aligned}$$

The desired bounds now follow readily from the inequalities

$$\begin{aligned} 0 \;\le\; (a-2)^2 \;\le\; 4\,; \qquad 0 \;\le\; \left(b-\frac{3}{2}\right)^2 \;\le\; \frac{9}{4}\,; \\ 0 \;\le\; (c-2)^2 \;\le\; 4\,; \qquad 0 \;\le\; \left(d-\frac{3}{2}\right)^2 \;\le\; \frac{9}{4}\,. \end{aligned}$$

18. A ray emanating from the vertex A of the triangle ABC intersects the side BC at X and the circumcircle of ABC at Y. Prove that

$$\frac{1}{AX}+\frac{1}{XY} \;\ge\; \frac{4}{BC}\,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's solution.

The proposed inequality can be rewritten as

$$BC\cdot AY \;\ge\; 4AX\cdot XY\,. \tag{1}$$

First, we suppose $AY<BC$. Since $XY=AY-AX$, the inequality (1) is equivalent to each of the following, the latter being obvious:

$$\begin{aligned} 4AX^2-4AX\cdot AY+BC\cdot AY &\;\ge\; 0\,, \\ (2AX-AY)^2+AY(BC-AY) &\;\ge\; 0\,. \end{aligned}$$

Now, suppose that $AY\ge BC$. Observing that $BC=XB+XC$ and $AX\cdot XY=XB\cdot XC$ (by the Intersecting Chord Theorem), inequality (1) may be written as

$$AY(XB+XC) \;\ge\; 4XB\cdot XC\,.$$

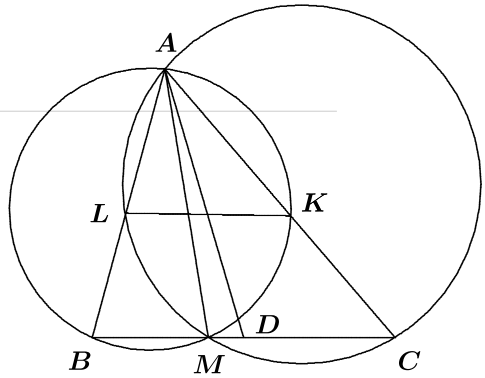
This inequality certainly holds, since

$$AY(XB+XC) \;\ge\; BC(XB+XC) \;=\; (XB+XC)^2 \;\ge\; 4XB\cdot XC\,.$$

19. In triangle ABC let D be the midpoint of BC and let M be a point on the side BC such that $\angle BAM=\angle DAC$. Let L be the second intersection point of the circumcircle of triangle CAM with AB, and let K be the second intersection point of the circumcircle of triangle BAM with the side AC. Prove that $KL\parallel BC$.

Solved by Oliver Geupel, Brühl, NRW, Germany; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comănești, Romania. We give Smeenk's solution.

Let $\alpha_1 = \angle BAM = \angle CAD$ and $\alpha_2 = \angle BAD = \angle CAM$, so $\alpha_1 + \alpha_2 = \alpha = \angle BAC$. Let $\beta = \angle ABC$, $\gamma = \angle ACB$. From the Sine Law in $\triangle ABM$ and $\triangle CAM$, we see that



$$AM = \frac{BM \sin\beta}{\sin\alpha_1} = \frac{CM \sin\gamma}{\sin\alpha_2}\,.$$

Therefore,

$$BM \sin\beta \sin\alpha_2 = CM \sin\gamma \sin\alpha_1\,. \quad (1)$$

Similarly, by the Sine Law in $\triangle BAD$ and $\triangle CAD$ we deduce that

$$BC \sin\beta \sin\alpha_1 = CD \sin\gamma \sin\alpha_2\,. \quad (2)$$

Now $BD = CD$, so (1), (2) imply $BM : CM = c^2 : b^2$, hence $CM = \dfrac{ab^2}{b^2+c^2}$.

As $CK \cdot CA = CM \cdot CB$, we find that $CK : b = a^2 : \left(b^2+c^2\right)$ and $BL : c = a^2 : \left(b^2+c^2\right)$, hence $KL \parallel BC$.

20. Three circular arcs w_1, w_2, and w_3 with common end-points A and B are on the same side of the line AB, and w_2 lies between w_1 and w_3. Two rays emanating from B intersect these arcs at M_1, M_2, M_3 and K_1, K_2, K_3, respectively. Prove that

$$\frac{M_1M_2}{M_2M_3} = \frac{K_1K_2}{K_2K_3}\,.$$

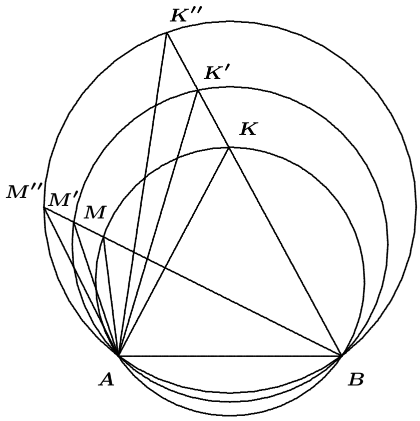
Solution by George Apostolopoulos, Messolonghi, Greece.

One less an index is so many dashes.

We have $\angle AKB = \angle AMB$ and $\angle AK'B = \angle AM'B$, because these are inscribed angles. It follows that $\triangle AKK'$ is similar to $\triangle AMM'$, and hence $\dfrac{KK'}{MM'} = \dfrac{AK'}{AM'}$.

Similarly, $\triangle AK'K''$ is similar to $\triangle AM'M''$, hence $\dfrac{K'K''}{M'M''} = \dfrac{AK'}{AM'}$.

Thus, $\dfrac{KK'}{MM'} = \dfrac{K'K''}{M'M''}$, and it follows that $\dfrac{MM'}{M'M''} = \dfrac{KK'}{K'K''}$, as desired.



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BOOK REVIEWS

Amar Sodhi

Polynomia and Related Realms: Uncommon Mathematical Excursions
By Dan Kalman, Dolciani Mathematical Expositions #35, Mathematical Association of America, 2009
ISBN 978-0-88385-341-2, cloth, 265+xv pages, US$61.95
Reviewed by ***Edward Barbeau****, University of Toronto, Toronto, ON*

By the seventeenth century, research into the systemic properties of polynomials and techniques for obtaining, approximating, and classifying their roots was well underway, and there is hardly a great mathematician of the last four hundred years who has not made a significant contribution in this area. Polynomials with real and complex coefficients is still an active area of mathematics, and there are lots of open problems. While many results are abstruse, technical, and very specialized, there is much to appeal to the general mathematician. The author of any book has a wealth of material to choose from for whatever audience is to be reached.

This book is a worthy addition to the Dolciani series. It is well written and contains material that is appealing (but perhaps not familiar) to every reader. However, it is not just about polynomials, which occupy only the first third of the book. The second part has to do with optimization while the third treats calculus. The author supplements the book by a website `http://www.maa.org/ume` in which he presents some additional material, in particular animated illustrations of topics in the book.

However, this is by no means a retail selling of standard topics. He casts the book as a travelogue, and decides not to go to the most popular and well-known sites, but to expand the visitor's experience by "seeking out lesser known destinations". While some students may enjoy the book, the author has written it more for their teachers who "have travelled through the standard curriculum" and "may wish to seek out new vistas and explore unsuspected wonders a bit off the beaten track". He also hopes that the book will be read by those who apply mathematics, such as scientists, engineers, and analysts. I am willing to bet that most readers will find something that is new to them.

The first destination is *The Province of Polynomia*. The trip opens with a brief prospectus and review of basic material. Chapter One discusses Horner's method for representing and evaluating polynomials; this is used to justify a geometric visualization of the roots of a polynomial due to an Austrian engineer, M.E. Lill, who published it in the *Nouvelles annales de mathématiques* in 1867. This method had been largely forgotten, and the author learned about it at a lecture by Tom Hull on origami (Hull's website is now at `http://mars.wnec.edu/~thull`, which supersedes the reference given in the book). The website for the book gives some dynamic illustrations of Lill's method.

The second chapter offers a fresh look at solving quadratic and cubic equations, and introduces Newton's method, Lagrange Interpolation and palindromic polynomials to the reader. It includes a discussion of Marden's theorem, to wit that, if $p(z)$ is a cubic complex polynomial with distinct non-collinear zeros, then the foci of the unique ellipse that touches the sides of the triangle determined by the zeros at their midpoints are the zeros of the derivative $p'(z)$; the website links to a proof. The third chapter treats roots, coefficients, and symmetric functions, while the fourth returns to equation solving.

The next port of call, *Maxministan*, focusses in particular on Lagrange multipliers. It is evident that the author has given a great deal of thought to what they signify and how they can be envisaged, so that this would be a valuable section for any undergraduate wanting to get beyond the usual cursory treatment. The multipliers are applied to a variety of situations, including the problem of finding the longest ladder that can be taken around a corner in a building and the problem of determining the maximum angle between the normal at a point of an ellipse and the line joining the point to the centre. There is a nice discussion of duality that relates a maximization problem to a corresponding minimization one. One topic that makes its first appearance here and is taken up in the last section of the book is that of an envelope of a family of curves.

Envelopes is a topic that used to be a common part of the tertiary syllabus but which has been absent for about the last half century. It is revisited in the *Calculusian Republic* where it is given a thorough treatment. However, this republic has other sites worthy of visit. Attention is focussed on equations of the form $a^x = mx+b$, where a, b, and m are parameters. Just as the logarithm function can be invented to solve the particular equation $a^x = b$, we can define the glog function in terms of which solutions of a more general form can be described; $y = \mathrm{glog}(x)$ if and only if $e^y = xy$ and the appropriate branch of the graph of the equation is taken. Properties and applications of this function are explored.

The book comes full circle back to polynomials and division by the monomial $x - a$ in its discussion of an algebraic approach to differentiation, "derivatives without limits". Glossing over questions of existence, this method can be extended to radicals, exponentials and trigonometric functions, where the treatment is incomplete for the last. The author concludes with a chapter giving a number of examples that evoke an appreciation of calculus, in particular its two miracles, *much out of little* and *more accuracy for less effort*.

There are several attractive features: essays in sidebars on particular topics, historical notes, and a section at the end of the chapter that provides historical background and a guide to the literature. There is a generous bibliography, with links provided on the author's website to electronic references. For lecturers in search of novel material and teachers interested in professional development, this book is highly recommended.

The Shape of Content: Creative Writings in Mathematics and Science
Edited by Chandler Davis, Marjorie Wikler Senechal, and Jan Zwicky, published by A.K. Peters, Ltd., 2008
ISBN 978-1-56881-444-5, hardcover, 194+xvii pages, US$39.00
Reviewed by **Georg Gunther**, *Sir Wilfred Grenfell College (MUN), Corner Brook, NL*

In 1959, C.P. Snow suggested that there had been a breakdown in communication between the "two cultures" of modern society – the sciences on the one hand, the humanities on the other. Since that time, this matter has been hotly contested by both sides and the debate continues to this day. Some argue that the differences between these "two cultures" can be traced down to the roots of our human creativity. They are seen as arising in part out of a biological asymmetry in the human brain, a lateralization of brain function, where the left hemisphere controls analytical and logical thought, while the right is the home of more holistic and creative mental activities.

The town of Banff is one of the great beauty spots in Canada, indeed, in the whole world. In addition to great scenery and spectacular skiing, Banff boasts of two world-class facilities. The first is the Banff Centre (founded in 1933), Canada's acclaimed artistic, cultural, and educational institute. The mission of the Banff Centre is expressed in two simple words: inspiring creativity. This mission, which had been limited to the arts and humanities, was expanded in the year 2000, when the Banff International Research Station for Mathematical Innovation and Discovery (BIRS) was established as part of the Banff Centre.

The first director of the BIRS was Robert Moody, not only a leading Canadian mathematician, but a serious photographer as well. Moody quickly realized that the presence of the BIRS on the campus of the Banff Centre provided a unique opportunity to bring together leading practitioners from both cultures in an attempt to see what sort of dialogue might develop. This idea led to a number of workshops on creative writing in mathematics and science; the first three of these were held in 2003, 2004, and 2006.

The book *The Shape of Content*, is a selection of contributions drawn from those three workshops, chosen because they "best conveyed the spirit, the meaning, and the achievements of the series". The contributing authors come from a wide range of disciplines; included in these are mathematics, biology, earth science, physics, chemistry, and philosophy. As well, there are pieces written by poets and playwrights, musicians and creative writers. The contributions themselves cover the spectrum of creative writing: prose pieces, biographical sketches, poetry, and excerpts of plays created during these workshops.

The Shape of Content is a wonderful book. It is not to be read in a single sitting, there are too many layers, too many subtleties. It needs to be sipped slowly, with the same appreciation that is due a very fine brandy. Nor is it to be read for its mathematical content; mathematics and science lurk in the backgrounds of many of the pieces, but their presence is largely metaphorical, as befits such powerful manifestations of our human creativity.

While the individual contributions differ greatly in the shape of their content, they all display "a crossing between the platonic world of ideas we mathematicians might be exploring, and the 'ideaspace' some philosophers (and some magicians) wonder about, a dimension inhabited by all the concepts humans (and aliens) could imagine, where all stories are true". [Marco Abarte, *Évariste and Héloise*, p. 5]. In *Active Pass* [p. 37], Isabel Burgess says "Not all things vanish into darkness. Some vanish into light"; so it is with many of the ideas, images, and phrases encountered in this volume.

Cosmologists speak of the Big Bang, a concept too grandiose, too staggering for us to comprehend. And yet, in his poem *The All of It* [p. 47], Robin Chapman makes it comprehensible, makes it human, when he writes "Still, the dark blue backdrop/ offers hope of god or natural law/ where beginnings are small enough/ to hold us all, the way the mind/ can hold the drinking glass/ or see the newborn child, that love/ set going from incomplete halves." As we ponder these words, our musings are enriched by Chandler Davis, when he writes, in the poem *Cold Comfort* [p. 53], "To fall back on predictability:/ All is caused, nothing will be forthcoming/ but what is embryonically already here,/ the mathematics tells truth about emergence".

The penultimate contribution to this volume is a description of a simple "kitchen-chemistry" experiment exploring the nature of soap. Single drops of food colour are added to a dish of milk; the colour droplets remain apart until a drop of soap is added; now "the isolated colours dance, play, and merge. Their yearning for community has been answered. They join together, ready to paint a picture of beauty and truth... All these miracles result from the simple fact that soap is able to operate at the same time in two worlds". [Randall Wedin, *Breaking Down the Barriers*, p. 182.]

Are there two cultures? Perhaps there are, but this volume is strong evidence that, no matter how far apart the sciences and the humanities might at times seem to be, bridges can be built, and when the effort is made to build them, the results can be spectacular, resulting in colours that dance, play, and merge. "Whatever is, is right./ This is not an order, but a riddle, not a single thought, but many", writes Adam Dickinson in his poem *Great Chain of Being* [p. 84]. So it is with *The Shape of Content*: not a single thought but many, many voices, singing together in harmony.

A Useful Inequality Revisited

Phạm Văn Thuận and Lê Vĩ

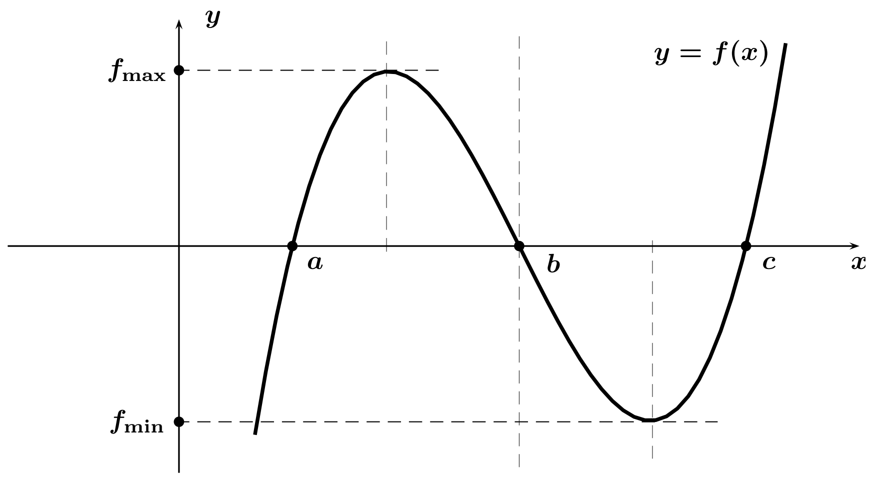
Abstract

We give a geometrical interpretation of a powerful inequality, and give some natural derivations of the inequality. We use this inequality to solve a large class of symmetric inequalities with four variables by using Rolle's theorem and some appropriate substitutions.

1 Introduction

Cẩn Võ Quốc Bà, [1] established a powerful inequality involving symmetric expressions, presumably stronger than the Schur Inequality. Later Roy Barbara [2] introduced another version of this inequality. However, it is not clear from [2] how the author could establish the set of coefficients for which the equality holds at two distinct points, say (k, k, k) or $(0, l, l)$ and its permutations, where k and l are nonnegative real numbers. Moreover, most problems in [2] are not strong enough to demonstrate the efficiency of this result. Herein we will give some interesting applications of this theorem.

Geometrically, if the x-intercepts of a cubic polynomial occur at distinct points a, b, and c, then its local maximum and local minimum values occur on opposite sides of the x-axis.



That is, the values of the cubic polynomial at these points are of opposite sign. This can be arrived at by noticing that the derivative has two distinct roots, in particular the discriminant of the derivative must be nonnegative.

We claim that the crux of the problem lies in this geometrical property of cubic polynomial functions. Indeed, the algebraic manipulations in the theorem below nicely reflect the geometric properties of the preceding figure.

Theorem 1 Let a, b, and c be nonnegative real numbers, let $p = a + b + c$, and suppose that $q = ab + bc + ca = \frac{p^2 - t^2}{3}$ for some $t \geq 0$. Then,

$$\frac{(p+t)^2(p-2t)}{27} \leq abc \leq \frac{(p-t)^2(p+2t)}{27}.$$

Proof: Let $f(x) = (x-a)(x-b)(x-c)$. We have

$$\begin{aligned} f(x) &= x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc\,, \\ f'(x) &= 3x^2 - 2(a+b+c)x + (ab+bc+ca)\,. \end{aligned}$$

The discriminant, Δ, of the quadratic polynomial $f'(x)$ is nonnegative, since

$$\begin{aligned} \Delta &= 4(a+b+c)^2 - 12(ab+bc+ca) \\ &= 2\big[(a-b)^2 + (b-c)^2 + (c-a)^2\big]\,. \end{aligned}$$

Alternatively, $\Delta = 4(p^2 - 3q) = 4t^2$, so that $\sqrt{\Delta} = 2t$ since $t \geq 0$. It follows that $f'(x)$ has the roots $c_1 = \frac{p-t}{3}$ and $c_2 = \frac{p+t}{3}$, and that $c_1 \leq c_2$.

If the roots a, b, and c of $f(x)$ are distinct or if two of these roots are equal but distinct from the remaining root, then by the usual methods of calculus for finding local extrema we find that $f_{\max} = f(c_1) \geq 0$ and $f_{\min} = f(c_2) \leq 0$. This last fact together with the computations

$$\begin{aligned} f_{\max} &= f(c_1) = f\left(\frac{p-t}{3}\right) = \frac{(p-t)^2(p+2t)}{27} - abc\,, \\ f_{\min} &= f(c_2) = f\left(\frac{p+t}{3}\right) = \frac{(p+t)^2(p-2t)}{27} - abc\,, \end{aligned}$$

gives us the desired inequality

$$\frac{(p+t)^2(p-2t)}{27} \leq abc \leq \frac{(p-t)^2(p+2t)}{27}.$$

If $a = b = c$, then the inequality is trivial. ∎

2 Applications

Theorem 1 has been employed to prove numerous inequalities of three variables, some of which are very hard, as shown in [1], [3]. The attempt to formulate an analogue of this inequality for four numbers, with the aim of proving a large class of four variable inequalities, has not yielded any fruitful results. In the following we shall provide some advances in this area.

Problem 1. Prove that if a, b, c, and d are nonnegative real numbers, then

$$\begin{aligned}(a+b+c+d)\left(a^2+b^2+c^2+d^2\right)^{3/2} \quad &\geq \\ \frac{1}{2}\left(a^3+b^3+c^3+d^3\right)(a+b+c+d) & \\ +\,(ab+bc+ca+da+db+dc)\left(a^2+b^2+c^2+d^2\right)\ . &\end{aligned}$$

Solution. It is a consequence of Rolle's theorem (note that if f is the monic quartic polynomial with roots a, b, c, and d, then f' has three real zeros) that there exist nonnegative numbers x, y, and z such that

$$\begin{aligned} x+y+z &= \frac{3}{4}(a+b+c+d)\,, \\ xy+yz+zx &= \frac{1}{2}(ab+bc+cd+da+ac+bd)\,, \\ xyz &= \frac{1}{4}(abc+bcd+cda+dab)\,. \end{aligned} \tag{1}$$

We also have the identities

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+b+c+d)^2-2(ab+bc+ca+da+db+dc)\,, \\ a^3+b^3+c^3+d^3 &= (a+b+c+d)^3+3(abc+bcd+cda+dab) \\ &= -\,3(ab+bc+ca+da+db+dc)(a+b+c+d)\,. \end{aligned}$$

Hence, we need to show that

$$\begin{aligned} \frac{4}{3}(x+y+z)\left(\frac{16}{9}(x+y+z)^2-4(xy+yz+zx)\right)^{3/2} \quad &\geq \\ \left(\frac{64}{27}(x+y+z)^3+12xyz-8(xy+yz+zx)(x+y+z)\right)\cdot\frac{2}{3}(x+y+z) & \\ +\,2(xy+yz+zx)\left(\frac{16}{9}(x+y+z)^2-4(xy+yz+zx)\right)\ . & \end{aligned}$$

For simplicity, we suppose without loss of generality that $x+y+z=1$. Let $q=xy+yz+zx$ and $r=xyz$. The inequality then reads

$$\frac{4}{3}\left(\frac{16}{9}-4q\right)^{3/2} \ \geq\ \frac{2}{3}\left(\frac{64}{27}+12r-8q\right)\ +\ 2q\left(\frac{16}{9}-4q\right)\ ,$$

which is equivalent to each of the following inequalities:

$$\begin{aligned} 12\left(\frac{16}{9}-4q\right)^{3/2} \quad &\geq \quad \frac{128}{9}+72r-48q+32q-72q^2\,, \\ 12\left(\frac{16}{9}-4q\right)^{3/2} \quad &\geq \quad \frac{128}{9}+72r-16q-72q^2\,. \end{aligned}$$

For $q=\dfrac{1-t^2}{3}$, $0\leq t\leq 1$, we have by Theorem 1 that $r\leq\dfrac{(1-t)^2(1+2t)}{27}$. Thus, it suffices to show that

$$\begin{aligned} 12\left(\frac{16}{9}-\frac{4}{3}\left(1-t^2\right)\right)^{3/2} \quad \geq \quad & \frac{128}{9}+\frac{8}{3}(1-t)^2(1+2t) \\ & -\frac{16}{3}\left(1-t^2\right)-8\left(1-t^2\right)^2\ . \end{aligned}$$

This is equivalent to each of the following

$$\begin{aligned} \frac{8}{9}\cdot 4\left(1+3t^2\right)^{3/2} &\ \geq\ \frac{32}{9}+\frac{40}{3}t^2+\frac{16}{3}t^3-8t^4\,,\\ 16\left(1+3t^2\right)^3-\left(4+6t^3+15t^2-9t^4\right)^2 &\ \geq\ 0\,,\\ 3t^2\left(8-16t+93t^2-60t^3+222t^4+36t^5-27t^6\right) &\ \geq\ 0\,, \end{aligned}$$

and the latter is clearly true for $0 \le t \le 1$.

Problem 2. Let a, b, c, and d be nonnegative real numbers which satisfy $a^2+b^2+c^2+d^2=1$. Prove that

$$\begin{aligned} (a+b+c+d)^2 \ \geq\ & a^3+b^3+c^3+d^3 \ +\ abc+bcd+cda+dab\\ & +(ab+bc+ca+da+db+dc)(a+b+c+d)\,. \end{aligned}$$

Solution. Write the desired inequality in homogenous form,

$$\begin{aligned} &(a+b+c+d)^2\left(a^2+b^2+c^2+d^2\right)^{1/2} \ \geq\\ &\quad a^3+b^3+c^3+d^3 \ +\ abc+bcd+cda+dab\\ &\quad +(ab+bc+ca+da+db+dc)(a+b+c+d)\,. \end{aligned}$$

We substitute as in (1) of the preceding solution and write $p=x+y+z$, $q=xy+yz+zx$, and $r=xyz$ to obtain the equivalent inequality

$$\frac{16}{9}p^2\left(\frac{16}{9}p^2-4q\right)^{1/2} \ \geq\ \frac{64}{27}p^3+12r-8pq+4r+\frac{8}{3}pq\,.$$

We group like terms and without loss of generality assume that $p=1$, so that the inequality reads

$$\frac{16}{9}\left(\frac{16}{9}-4q\right)^{1/2} \ \geq\ \frac{64}{27}+16r-\frac{16}{3}q\,.$$

Let $q=\dfrac{1-t^2}{3}$, $0\le t\le 1$. By Theorem 1, we have that $r \le \dfrac{(1-t)^2(1+2t)}{27}$. Thus, it suffices to show that

$$\frac{16}{9}\left(\frac{16}{9}-\frac{4}{3}\left(1-t^2\right)\right)^{1/2} \ \geq\ \frac{64}{27}+\frac{16}{27}(1-t)^2(1+2t)-\frac{16}{9}\left(1-t^2\right)\,.$$

After some algebraic manipulations, this is equivalent to $1+3t^2 \ge \left(1+t^3\right)^2$, or $t^2\left(3-2t-t^4\right)\ge 0$, which is obvious since $0\le t\le 1$. ■

It is impossible to reduce an inequality involving the product of four variables to a three-variable inequality using Rolle's theorem. Without Rolle's theorem, the idea is to reduce the number of variables by exploiting the homogeneity and arranging the given variables.

Problem 3 (János Surányi's inequality). Prove that if a, b, c, and d are nonnegative real numbers, then

$$3\left(a^4+b^4+c^4+d^4\right)+4abcd \ \geq\ (a+b+c+d)\left(a^3+b^3+c^3+d^3\right)\,.$$

Solution If one of a, b, c, or d is zero, then the desired inequality is true (for instance, one may cancel terms and apply the Rearrangement Inequality twice). If a, b, c, and d are all positive, then (due to homogeneity) we can suppose without loss of generality that

$$d \;=\; \min\{a,b,c,d\} \;=\; 1\,. \tag{2}$$

The desired inequality becomes

$$3\left(a^4+b^4+c^4+1\right)+4abc \;\geq\; (a+b+c+1)\left(a^3+b^3+c^3+1\right)\,. \tag{3}$$

Let $p=a+b+c$, $q=ab+bc+ca$, and $r=abc$. By the assumption we made in (2), we have $p \geq 3$. Since

$$\begin{aligned} a^4+b^4+c^4 &= \left(p^2-2q\right)^2-2\left(q^2-2pr\right)\,, \\ a^3+b^3+c^3 &= p\left(p^2-3q\right)+3r\,, \end{aligned}$$

the inequality (3) takes the form

$$2p^4-p^3-p+2-9p^2q+3pq+6q^2+9pr+r \;\geq\; 0\,. \tag{4}$$

By Theorem 1,

$$r \;\geq\; \frac{1}{27}\left(p^3-3pt^2-2t^3\right)\,.$$

Therefore, it suffices to show that

$$\begin{aligned} &2p^4-p^3-p+2-9p^2\left(\frac{p^2-t^2}{3}\right)+3p\left(\frac{p^2-t^2}{3}\right) \\ &+6\cdot\frac{\left(p^2-t^2\right)^2}{9}+\frac{p}{3}\left(p^3-3pt^2-2t^3\right)+\frac{1}{27}\left(p^3-3pt^2-2t^3\right) \;\geq\; 0\,, \end{aligned}$$

which is equivalent to

$$(p-3)^2(p+6)+2t^2\left(9p^2-15p-9pt+9t^2-t\right) \;\geq\; 0\,. \tag{5}$$

Now $3p^2+9t^2 \geq 9pt$ and furthermore since $p \geq 3$ and $p \geq t$, we have $6p^2 \geq 18p > 15p+t$. It follows that

$$9p^2-15p-9pt+9t^2-t \;\geq\; 0\,,$$

hence (5) is true and the inequality is proved. ∎

Problem 4 (IMO Shortlist 1993). Prove that if a, b, c, and d are nonnegative integers, then

$$(a+b+c+d)^4+176abcd \;\geq\; 27(a+b+c+d)(abc+bcd+cda+dab)\,.$$

Solution. If one of the four numbers is zero, then the inequality follows from the AM–GM Inequality. Otherwise we suppose (due to homogeneity) as in (2) above that $d=\min\{a,\,b,\,c,\,d\}=1$. We then need to prove that

$$(a+b+c+1)^4+176abc \;\geq\; 27(a+b+c+1)(abc+ab+ac+bc)\,.$$

Let $p = a+b+c$, $q = ab+bc+ca$, and $r = abc$. From our assumption we have $p \geq 3$ and $q \geq 3$, and the desired inequality takes the form

$$p^4 + 4p^3 + 6p^2 + 4p + 1 - 27pq - 27q + (149 - 27p)r \;\geq\; 0\,. \qquad (6)$$

For $q = \frac{1}{3}(p^2 - t^2)$, $0 \leq t \leq p$, we have by Theorem 1 that

$$\frac{(p+t)^2(p-2t)}{27} \;\leq\; r \;\leq\; \frac{(p-t)^2(p+2t)}{27}\,.$$

If $3 \leq p \leq \dfrac{149}{27}$, then $149 - 27p \geq 0$. In this case, by Theorem 1, in order to prove (6) it suffices for us to show that

$$\frac{1}{27}(14p+3)(p-3)^2 + t^2\left(3p^2 - \frac{68}{9}p + 9 - \frac{298}{27}t + 2pt\right) \;\geq\; 0\,.$$

The inequality

$$3p^2 - \frac{68}{9}p + 9 - \frac{298}{27}t + 2pt \;\geq\; 0$$

is proved by adding across the following four inequalities

$$p^2 + 9 \geq 6p\,; \qquad \frac{14}{27}p^2 \geq \frac{14}{9}p\,; \qquad 2pt \geq 6t\,; \qquad \frac{40}{27}p^2 \geq \frac{136}{27}t\,.$$

The last inequality is valid since it follows from $q = \frac{1}{3}(p^2 - t^2) \geq 3$ that $p^2 \geq t^2 + 9 \geq 6t$.

If $149 - 27p \leq 0$, then to prove (6) it suffices for us to show that

$$\frac{1}{27}(14p+3)(p-3)^2 + t^2\left(3p^2 - \frac{68}{9}p + 9 + \frac{298}{27}t - 2pt\right) \;\geq\; 0\,.$$

Consider the function

$$f(p) \;=\; 3p^2 - \frac{68}{9}p + 9 + \frac{298}{27}t - 2pt\,.$$

We have

$$f'(p) \;=\; 6p - \frac{68}{9} - 2t\,.$$

Since $p \geq t$ and $p \geq \frac{149}{27}$, then $f'(p) > 0$. Thus, $f(p)$ is increasing for $p \geq \max\left\{t, \frac{149}{27}\right\}$. It follows that

$$f(p) \;\geq\; f\left(\max\left\{t, \frac{149}{27}\right\}\right) \;\geq\; 0\,,$$

and the inequality is proved. ■

Problem 5. Let m, n, u, and v be real numbers such that all of the roots of

$$x^4 - mx^3 + nx^2 - ux + v \;=\; 0\,.$$

are nonnegative. Prove that $m^4 + 32v \geq 3m^2n$.

Solution. Given m, n, u, and v as above, let a, b, c, and d be nonnegative real numbers such that $x^4 - mx^3 + nx^2 - ux + v = (x-a)(x-b)(x-c)(x-d)$. Then we need to prove that

$$(a+b+c+d)^4 + 32abcd \;\geq\; 3(a+b+c+d)^2(ab+bc+ca+ad+bd+cd)\,. \quad (7)$$

The problem can be restated as follows: Prove that if four nonnegative real numbers a, b, c, and d satisfy $a+b+c+d=1$, then

$$1 + 32abcd \;\geq\; 3(ab+bc+ca+ad+bd+cd)\,.$$

In this form of the problem we see that if one of the four numbers is zero, then the desired inequality follows immediately from the Cauchy Inequality.

Thus, we can suppose that a, b, c, and d are all positive, and furthermore we can suppose (due to homogeneity) that

$$d \;=\; \min(a,b,c,d) \;=\; 1\,. \quad (8)$$

The inequality (7) then takes the form

$$(a+b+c+1)^4 + 32abc \;\geq\; 3(a+b+c+1)^2(ab+bc+ca+a+b+c)\,.$$

Let $p = a+b+c$, $q = ab+bc+ca$, and $r = abc$. By our assumption (8) we have that $p \geq 3$ and $q \geq 3$. In terms of p, q, and r the inequality becomes

$$(p+1)^4 + 32r \;\geq\; 3(p+1)^2(q+p)\,.$$

Expanding this yields

$$p^4 + p^3 + p + 1 - 3p^2q - 6pq - 3q + 32r \;\geq\; 0\,.$$

Let $q = \frac{1}{3}(p^2 - t^2)$, $0 \leq t \leq p$. By Theorem 1 we have that

$$r \;\geq\; \frac{1}{27}(p+t)^2(p-2t) \;=\; \frac{1}{27}\left(p^3 - 3pt^2 - 2t^3\right)\,.$$

Thus, it suffices to show that

$$p^4 + p^3 + p + 1 - (p^2+2p+1)\left(p^2 - t^2\right) + \frac{32}{27}\left(p^3 - 3pt^2 - 2t^3\right) \;\geq\; 0\,,$$

which is equivalent to

$$\frac{1}{27}(5p+3)(p-3)^2 + t^2\left(p^2 + 1 - \frac{14}{9}p - \frac{64}{27}t\right) \;\geq\; 0\,. \quad (9)$$

Moreover, since $p \geq 3$ we have $\frac{14}{27}p^2 \geq \frac{14}{9}p$ and since $q = \frac{1}{3}(p^2 - t^2) \geq 3$ we have $p^2 \geq t^2 + 9$. Consequently,

$$\frac{13}{27}p^2 + 1 \;\geq\; \frac{13}{27}t^2 + \frac{16}{3} \;\geq\; \frac{64}{27}t\,.$$

It follows that

$$p^2 + 1 - \frac{14}{9}p - \frac{64}{27}t \;\geq\; 0$$

and hence the inequality (9) is true. Equality occurs when (a,b,c,d) is the vector $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ or some permutation of the vector $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. ■

3 Exercises

Problem 6. Let x, y, z, and w be nonnegative real numbers. Prove that

$$\left(x^4+y^4+z^4+w^4\right)(xy+yz+zx+wx+wy+wz) \le \frac{3}{8}\left(x^2+y^2+z^2+w^2\right)^3.$$

Problem 7. Let a, b, c, and d be nonnegative real numbers. Prove that

$$\begin{aligned}\frac{1}{2}\left(a^2+b^2+c^2+d^2\right)^{5/2}+2abcd(a+b+c+d) \quad &\ge \\ (ab+bc+cd+da+ac+bd)&\left(a^3+b^3+c^3+d^3\right) .\end{aligned}$$

Problem 8. Let a, b, c, and d be nonnegative real numbers. Prove that

$$a^3+b^3+c^3+d^3 \; + \; \frac{32abcd}{a+b+c+d} \;\ge\; 3(abc+bcd+cda+dab) .$$

Problem 9. Let a, b, c, and d be nonnegative real numbers which satisfy $a^2+b^2+c^2+d^2=1$. Prove that

$$a+b+c+d \;\ge\; a^3+b^3+c^3+d^3 \;+\; ab+bc+ca+cd+da+bd .$$

Problem 10. Let m, n, u, and v be real numbers such that all zeros of the quartic polynomial $x^4-mx^3+nx^2-ux+v$ are nonnegative real numbers. Prove that

$$\left(m^2-2n\right)^{5/2}+8mv \;\ge\; 4\left(m^2-2n\right)u .$$

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 October 2009**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3426. *Proposed by Salvatore Tringali, student, Mediterranea University, Reggio Calabria, Italy.*

Find all prime numbers p, q, and r such that $p+q=(p-q)^r$.

3427. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The numbers a, b, c, and d all lie in the interval $(1,\infty)$ and are such that $a+b+c+d=16$. Prove that

$$\sum_{\text{cyclic}} \log_a\left(\sqrt[4]{bcd}+a\right) \ \geq\ \frac{11}{2}\,.$$

3428★. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Fix an integer $n>2$ and let I be the interval of all positive ratios r such that there exists an n-gon whose sides consist of n terms of a geometric sequence with common ratio r. Prove that the endpoints of I are reciprocals of each other.

[*Ed.*: The proposer refers to ***Crux*** M67 [2003 : 430-431] and 3082 [2006 : 477] for the special cases $n=3$ and $n=4$.]

3429. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

The line ℓ passes through the point A and makes an acute angle with the segment AB. The line m passes through B and is perpendicular to AB. Construct a point C on the line ℓ and a point P on the line m such that the triangle BPC is isosceles with $BP=PC$ and

(a) the line CP trisects $\angle BCA$,

(b) the line CP bisects $\angle BCA$.

3430. *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Determine the coefficients of the unique polynomial $P_n(x)$ for which the relation

$$\cos^{2n}\theta + \sin^{2n}\theta \;=\; P_n\left(\sin^2(2\theta)\right)$$

holds for all real numbers θ.

3431. *Proposed by Michel Bataille, Rouen, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies

$$f(x+y) \;=\; f\bigl(f(x)\cdot f(y)\bigr)$$

for all real numbers x and y. Prove that f is constant.

3432. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be real numbers satisfying $a < 2(b+c)$, $b < 2(c+a)$, and $c < 2(a+b)$. Prove that

$$3 \;\le\; \frac{4\left(a^3+b^3+c^3\right)+15abc}{(a+b+c)(ab+bc+ca)} \;<\; 6\,.$$

3433. *Proposed by an unknown proposer.*

For each positive integer n prove that

$$\sum_{k=0}^{n}\frac{1}{2k+1}\binom{2k}{k}\binom{2n-2k}{n-k} \;=\; \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}.$$

3434. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given the line segment LMN with $LM : MN = 1 : \lambda$ and $\lambda > 0$, and given the triangle ABC with $\angle ABC = x + y$ and $\dfrac{\tan x}{\tan y} = \dfrac{1}{\lambda}$, construct the angle x using only a straight edge and compass.

3435. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let a, b, c, and d be positive integers. Prove that

$$\frac{1}{a+b+c+d+2} - \frac{1}{(a+1)(b+1)(c+1)(d+1)} \;\le\; \frac{5}{48}\,.$$

3436. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbie.*

Let ABC be a right-angled triangle with hypotenuse AB. Let m_a and m_b be the lengths of the medians to the sides BC and AC, respectively. Prove that

$$\frac{\sqrt{5}}{2} \le \frac{m_a + m_b}{a+b} < \frac{3}{2}.$$

3437. *Proposed by Pham Huu Duc, Ballajura, Australia and Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a^2+bc}{b+c}} + \sqrt{\frac{b^2+ca}{c+a}} + \sqrt{\frac{c^2+ab}{a+b}} \ge \sqrt{3(a+b+c)}.$$

3438★. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be nonnegative real numbers. Prove the inequality below for all $\kappa \ge 0$, or give a counterexample:

$$\sum_{\text{cyclic}} \sqrt{\frac{a^2+\kappa bc}{b^2+c^2}} \ge 2 + \sqrt{\frac{\kappa}{2}}.$$

. . . .

3426. *Proposé par Salvatore Tringali, étudiant, Université Mediterranea, Reggio Calabria, Italie.*

Trouver tous les nombres premiers p, q et r tels que $p+q=(p-q)^r$.

3427. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Les nombres a, b, c et d sont tous dans l' intervalle $(1,\infty)$ et sont tels que $a+b+c+d=16$. Montrer que

$$\sum_{\text{cyclique}} \log_a\left(\sqrt[4]{bcd}+a\right) \ge \frac{11}{2}.$$

3428★. *Proposé par J. Walter Lynch, Athens, GA, É-U.*

On fixe un entier $n > 2$ et soit I l'intervalle de tous les quotients positifs r tel qu'il existe un n-gone dont les côtés consistent en n termes d'une suite géométrique de raison r. Montrer que les extrémités de I sont réciproques l'un de l'autre.

[N.d.R : Pour les cas spéciaux $n=3$ et $n=4$, le proposeur renvoie à Crux M67 [2003 :430-431] et 3082 [2006 :477]].

3429. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

La droite ℓ passe par le point A et fait un angle aigu avec le segment AB. La droite m passe par B et est perpendiculaire à AB. Construire un point C sur la droite ℓ et un point P sur la droite m de sorte que le triangle BPC soit isocèle avec $BP = PC$ et que

(a) la droite CP soit une trisection de l'angle BCA,

(b) la droite CP soit la bissectrice de l'angle BCA.

3430. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier positif. Déterminer les coefficients de l'unique polynôme $P_n(x)$ pour lequel la relation

$$\cos^{2n}\theta + \sin^{2n}\theta \;=\; P_n\left(\sin^2(2\theta)\right)$$

est satisfaite pour tous les nombres réels θ.

3431. *Proposé par Michel Bataille, Rouen, France.*

Soit $f : \mathbb{R} \to \mathbb{R}$ une fonction continue satisfaisant

$$f(x+y) \;=\; f\big(f(x)\cdot f(y)\big)$$

pour tous les nombres réels x et y. Montrer que f est constante.

3432. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c trois nombres réels satisfaisant $a < 2(b+c)$, $b < 2(c+a)$ et $c < 2(a+b)$. Montrer que

$$3 \;\le\; \frac{4\left(a^3+b^3+c^3\right)+15abc}{(a+b+c)(ab+bc+ca)} \;<\; 6\,.$$

3433. *Proposé par un proposeur anonyme.*

Pour tout entier positif n, montrer que

$$\sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\binom{2n-2k}{n-k} \;=\; \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}.$$

3434. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Sur une droite, on donne le segment LMN avec $LM : MN = 1 : \lambda$ et $\lambda > 0$, et on considère le triangle ABC avec l'angle $ABC = x + y$ et $\dfrac{\tan x}{\tan y} = \dfrac{1}{\lambda}$. On demande de construire l'angle x en n'utilisant que la règle et le compas.

3435. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit a, b, c et d quatre entiers positifs. Montrer que

$$\frac{1}{a+b+c+d+2} - \frac{1}{(a+1)(b+1)(c+1)(d+1)} \le \frac{5}{48}\,.$$

3436. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit ABC un triangle rectangle d'hypoténuse AB. Soit respectivement m_a et m_b les longueurs des médianes aboutissant sur les côtés BC et AC. Montrer que

$$\frac{\sqrt{5}}{2} \le \frac{m_a+m_b}{a+b} < \frac{3}{2}\,.$$

3437. *Proposé par Pham Huu Duc, Ballajura, Australie and Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sqrt{\frac{a^2+bc}{b+c}} + \sqrt{\frac{b^2+ca}{c+a}} + \sqrt{\frac{c^2+ab}{a+b}} \ge \sqrt{3(a+b+c)}\,.$$

3438★. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Soit a, b et c trois nombres réels non négatifs. Montrer la validité de l'inégalité ci-dessous pour tout $\kappa \ge 0$, ou donner un contre-exemple :

$$\sum_{\text{cyclique}} \sqrt{\frac{a^2+\kappa bc}{b^2+c^2}} \ge 2 + \sqrt{\frac{\kappa}{2}}\,.$$

Fib!
Lie!
Tell it!
Who is it
Made this story up?
Can you find the answer to this?
There is glory waiting for the correct solution!

(This is a Fibonacci poem!

Search for "pincus+fibonacci" on the web to find out more.)

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Last year we received a batch of correct solutions from Steven Karp, student, University of Waterloo, Waterloo, ON, to problems 3289, 3292, 3294, 3296, 3297, 3298, and 3300, which did not make it into the December issue due to being misfiled. Our apologies for this oversight.

3326. [2008 : 170, 173] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers.

(a) Show that $\prod\limits_{\text{cyclic}} (a^2+2) + 4\prod\limits_{\text{cyclic}} (a^2+1) \ \geq\ 6(a+b+c)^2$.

(b)★ What is the largest constant k such that

$$\prod_{\text{cyclic}} (a^2+2) + 4\prod_{\text{cyclic}} (a^2+1) \ \geq\ k(a+b+c)^2\,?$$

The same solution to part (a) *by George Apostolopoulos, Messolonghi, Greece and the proposer, modified and expanded by the editor.*

Let $P_1 = \prod\limits_{\text{cyclic}} (a^2+2)$ and $P_2 = \prod\limits_{\text{cyclic}} (a^2+1)$. We first prove that

$$P_1 \ \geq\ 3(a+b+c)^2\,, \tag{1}$$

or equivalently

$$\prod_{\text{cyclic}} \left(1 + \frac{a^2-1}{3}\right) \ \geq\ \frac{(a+b+c)^2}{9}\,. \tag{2}$$

We consider three cases:

Case 1 At least two of a, b, and c are at least 1.

Say $a \geq 1$ and $b \geq 1$. Then since $(1+x)(1+y) \geq 1+x+y$ for $x \geq 0$ and $y \geq 0$, we have

$$\begin{aligned}\prod_{\text{cyclic}} \left(1 + \frac{a^2-1}{3}\right) &\ \geq\ \left(1 + \frac{a^2-1}{3} + \frac{b^2-1}{3}\right)\left(\frac{c^2+2}{3}\right)\\ &=\ \left(\frac{a^2+b^2+1^2}{3}\right)\left(\frac{1^2+1^2+c^2}{3}\right) \ \geq\ \frac{(a+b+c)^2}{9}\end{aligned}$$

by the Cauchy–Schwarz Inequality.

Case 2 One of a, b, or c is at least 1.

Say $a \geq 1$, $b < 1$, and $c < 1$. We have $(1+x)(1+y) \geq 1+x+y$ if x and y are both in the interval $(-1,0)$. Taking $x = \dfrac{b^2-1}{3}$ and $y = \dfrac{c^2-1}{3}$ we then obtain

$$\begin{aligned}\prod_{\text{cyclic}} \left(1 + \frac{a^2-1}{3}\right) \;&\geq\; \left(1+\frac{a^2-1}{3}\right)\left(1+\frac{b^2-1}{3}+\frac{c^2-1}{3}\right) \\ &=\; \left(\frac{a^2+1^2+1^2}{3}\right)\left(\frac{1^2+b^2+c^2}{3}\right) \;\geq\; \frac{(a+b+c)^2}{9}\,.\end{aligned}$$

Case 3 Each of a, b, and c is less than 1.

We have $(1+x)(1+y)(1+z) \geq 1+x+y+z$ if x, y, and z are in $(-1,0)$. Hence, since $\dfrac{a^2-1}{3}$, $\dfrac{b^2-1}{3}$, and $\dfrac{c^2-1}{3}$ each lie in the interval $(-1,0)$ we have

$$\begin{aligned}&\prod_{\text{cyclic}} \left(1 + \frac{a^2-1}{3}\right) \;\geq\; 1 + \frac{a^2-1}{3} + \frac{b^2-1}{3} + \frac{c^2-1}{3} \\ =\;& \frac{a^2+b^2+c^2}{3} \;=\; \frac{1}{9}\left(a^2+b^2+c^2\right)\left(1^2+1^2+1^2\right) \;\geq\; \frac{(a+b+c)^2}{9}.\end{aligned}$$

Therefore, equation (2) holds in all cases, and the proof of equation (1) is complete.

Next we replace a, b, and c in equation (1) by $\sqrt{2}a$, $\sqrt{2}b$, and $\sqrt{2}c$, respectively. Then (1) becomes $8 \prod\limits_{\text{cyclic}} \left(a^2+1\right) \geq 6(a+b+c)^2$ or

$$4P_2 \;\geq\; 3(a+b+c)^2\,. \tag{3}$$

The conclusion now follows by adding (1) and (3).

Part (a) *also solved by MICHEL BATAILLE, Rouen, France; and CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA.*

Oliver Geupel, Brühl, NRW, Germany pointed out that part (a) *follows from Problem 3327, part* (a).

No complete solution to part (b) *was received. Both Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Stan Wagon, Macalester College, St. Paul, MN, USA produced complicated expressions for the constant k in part* (b). *Wagon obtained his "in a moment" using Mathematica's Maximize Function and evaluated $k = 6.24347$ to 5 decimal places. Janous obtained a similar expression and evaluated $k = 6.243471387$ to 9 decimal places. However, he stated his result as a conjecture, since he assumed in the course of his calculations that the ratio $(P_1 + 4P_2)/(a+b+c)^2$ is minimized when $a = b = c$. The editor strongly suspects these answers are correct, but has received only incomplete arguments of their validity.*

3327. [2008 : 170, 173] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers.

(a) Show that $\prod_{\text{cyclic}} (a^4 + 3a^2 + 2) \;\geq\; \frac{9}{4}(a+b+c)^4$.

(b)★ What is the largest constant k such that

$$\prod_{\text{cyclic}} (a^4 + 3a^2 + 2) \;\geq\; k(a+b+c)^4\,?$$

Similar solutions by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

First, observe that $a^4 + 3a^2 + 2 = \left(a^2+2\right)\left(a^2+1\right)$. In the solution of the preceding problem the inequality

$$\prod_{\text{cyclic}} \left(a^2+2\right) \;\geq\; 3(a+b+c)^2$$

and the inequality

$$\prod_{\text{cyclic}} \left(a^2+1\right) \;\geq\; \frac{3}{4}(a+b+c)^2$$

was proven. The result now follows by multiplying across these two inequalities.

Part (a) *was also solved by* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and the proposer. There was one incomplete solution submitted. Part* (b) *remains open.*

3328★. [2008 :170, 172] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. For $1 \leq k \leq n$, define

$$A_k \;=\; \frac{1}{k}\sum_{i=1}^{k} a_i\,, \quad G_k \;=\; \left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k}}, \quad \text{and} \quad H_k \;=\; k\left(\sum_{i=1}^{k} \frac{1}{a_i}\right)^{-1}.$$

(a) Show that $\frac{1}{n}\sum\limits_{k=1}^{n} G_k \;\leq\; \left(\prod\limits_{k=1}^{n} A_k\right)^{\frac{1}{n}}$.

(b) Show that $n\left(\sum\limits_{k=1}^{n} \frac{1}{G_k}\right)^{-1} \;\geq\; \left(\prod\limits_{k=1}^{n} H_k\right)^{\frac{1}{n}}$.

Similar solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.

The inequality in part (a) is known: see Kiran S. Kedlaya, "Proof of a mixed arithmetic-mean, geometric-mean inequality", *Amer. Math. Monthly*, Vol. 101, No. 4 (1994), pp. 355–357.

To prove part (b), we replace the numbers a_1, a_2, $\ldots$, a_n in part (a) with their reciprocals $\frac{1}{a_1}$, $\frac{1}{a_2}$, $\ldots$, $\frac{1}{a_n}$ and for each k we let A'_k and G'_k be the resulting arithmetic and geometric means. Clearly, $G'_k = \frac{1}{G_k}$ and $A'_k = \frac{1}{H_k}$. By part (a) we have

$$\frac{1}{n}\sum_{k=1}^{n} G'_k \;\le\; \left(\prod_{k=1}^{n} A'_k\right)^{\frac{1}{n}} ,$$

therefore,

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{G_k} \;\le\; \left(\prod_{k=1}^{n} \frac{1}{H_k}\right)^{\frac{1}{n}} \;=\; \left(\prod_{k=1}^{n} H_k\right)^{-\frac{1}{n}}$$

and

$$n\left(\sum_{k=1}^{n} \frac{1}{G_k}\right)^{-1} \;\ge\; \left(\prod_{k=1}^{n} H_k\right)^{\frac{1}{n}} ,$$

as required.

Part (a) *also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and OLIVER GEUPEL, Brühl, NRW, Germany.*

Curtis cites the reference in our featured solution and in addition he cites Takashi Matsuda, "An inductive proof of a mixed arithmetic–geometric mean inequality", Amer. Math. Monthly*, Vol. 102, No. 7 (1995), pp. 634–637. He informs us that Kedlaya's proof is combinatorial in nature, while Matsuda's proof uses induction and Lagrange Multipliers.*

3329. [2008 : 171, 173] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r be a real number, $0 < r \le 1$, and let x, y, and z be positive real numbers such that $xyz = r^3$. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \;\le\; \frac{3}{\sqrt{1+r^2}} .$$

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

First we prove that for $0 < s \le 1$ and x, $y > 0$ such that $xy = s^2$,

$$\frac{1}{\sqrt{1+x^2}} \;+\; \frac{1}{\sqrt{1+y^2}} \;\le\; \frac{2}{\sqrt{1+s^2}} . \qquad (1)$$

Let $\alpha = \arctan x$ and $\beta = \arctan y$. Since $\tan\alpha\tan\beta = xy = s^2 \leq 1$, we have $\alpha + \beta \leq \frac{\pi}{2}$. Thus $\tan\alpha\tan\beta \leq \tan^2\left(\frac{\alpha+\beta}{2}\right)$ (see the book by Ivan Niven and Lester H. Lance, *Maxima and Minima Without Calculus*, p. 103). Therefore

$$\begin{aligned}
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
&\leq 2\cos\left(\frac{\alpha+\beta}{2}\right) = \frac{2}{\sqrt{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)}} \\
&\leq \frac{2}{\sqrt{1+\tan\alpha\tan\beta}}\,.
\end{aligned}$$

This inequality implies (1). For $xyz = r^3$, we have $\min\{xy,\, yz,\, zx\} \leq 1$. We may assume (by symmetry) that $xy \leq 1$. Set $xy = s^2$. By the previous result

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+s^2}}$$

and to obtain the given inequality it is enough to prove that if $z > 0$ and $s^2 z = r^3$, then

$$\frac{2}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{\sqrt{1+r^2}}\,. \tag{2}$$

For $z > 0$ let

$$f(z) = \frac{1}{\sqrt{1+z^2}} + \frac{2}{\sqrt{1+\dfrac{r^3}{z}}} = \frac{1}{\sqrt{1+z^2}} + \frac{2\sqrt{z}}{\sqrt{z+r^3}}\,.$$

Direct computation gives $f'(z) = -\frac{z}{(1+z^2)^{3/2}} + \frac{r^3}{\sqrt{z}(z+r^3)^{3/2}}$. Further calculations reveal that $f'(z) = 0$ if and only if $(z-r)\big((1-r^2)z + r\big) = 0$, hence $z = r$ is the only (positive) zero of f'. Since $\lim\limits_{z\to 0^+} f(z) = 1$ and we also have $f(r) = \frac{3}{\sqrt{1+r^2}} \geq \frac{3}{\sqrt{2}} > 2$, it follows that f has an absolute maximum at $z = r$ and the inequality (2) holds.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; ADAM STRZEBONSKI, Wolfram Research Inc., Champaign, IL, USA and STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incorrect solutions submitted.

3331. [2008 : 171, 174] *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be the lengths of the sides of triangle ABC, and let R be its circumradius. Prove that

$$\sqrt[3]{a^2b}+\sqrt[3]{b^2c}+\sqrt[3]{c^2a} \;\le\; 3\sqrt{3}\,R\,.$$

Solution by Mihály Bencze, Brasov, Romania.

We prove the following generalization: If x, y, and z are (fixed) positive real numbers, then

$$\sum_{\text{cyclic}}\left(a^xb^yc^z\right)^{\frac{1}{x+y+z}} \;\le\; 3\sqrt{3}R\,.$$

Using the weighted AM–GM Inequality and the well-known inequality $a+b+c \le 3\sqrt{3}R$ (item 5.3 in *Geometric Inequalities* by O. Bottema et al., Groningen, 1969), we obtain

$$\sum_{\text{cyclic}}\left(a^xb^yc^z\right)^{\frac{1}{x+y+z}} \;\le\; \sum_{\text{cyclic}}\frac{ax+by+cz}{x+y+z} \;=\; a+b+c \;\le\; 3\sqrt{3}R\,,$$

which completes the proof. Taking $x=2$, $y=1$, and $z=0$ yields the desired inequality.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposers.

3332. [2008 : 171, 174] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a_1, a_2, a_3, and a_4 be positive real numbers and let λ and μ be positive integers.

(a) Prove that

$$\frac{a_1}{\lambda a_2+\mu a_3}+\frac{a_2}{\lambda a_3+\mu a_1}+\frac{a_3}{\lambda a_1+\mu a_2} \;\ge\; \frac{3}{\lambda+\mu}\,.$$

(b) Prove that

$$\frac{a_1}{\mu a_2 + \mu a_3 + \mu a_4} + \frac{a_2}{\lambda a_3 + \lambda a_4 + \lambda a_1} + \frac{a_3}{\mu a_4 + \lambda a_1 + \mu a_2} + \frac{a_4}{\lambda a_1 + \mu a_2 + \lambda a_3} \geq \frac{8}{3(\lambda + \mu)}.$$

Similar solutions by Mihály Bencze, Brasov, Romania and George Tsapakidis, Agrinio, Greece.

Let λ and μ be positive real numbers.

(a) We have

$$\begin{aligned}
&\frac{a_1}{\lambda a_2 + \mu a_3} + \frac{a_2}{\lambda a_3 + \mu a_1} + \frac{a_3}{\lambda a_1 + \mu a_2} \\
&\quad = \frac{a_1^2}{\lambda a_2 a_1 + \mu a_3 a_1} + \frac{a_2^2}{\lambda a_3 a_2 + \mu a_1 a_2} + \frac{a_3^3}{\lambda a_1 a_3 + \mu a_2 a_3} \\
&\quad \geq \frac{(a_1 + a_2 + a_3)^2}{(\lambda + \mu)(a_1 a_2 + a_2 a_3 + a_3 a_1)} \geq \frac{3}{\lambda + \mu},
\end{aligned}$$

since $(a_1 + a_2 + a_3)^2 \geq 3(a_1 a_2 + a_2 a_3 + a_3 a_1)$. [*Ed.: The Cauchy–Schwarz Inequality is used for the first inequality in the above display; multiplying each side by* $(\lambda a_2 a_1 + \mu a_3 a_1) + (\lambda a_3 a_2 + \mu a_1 a_2) + (\lambda a_1 a_3 + \mu a_2 a_3)$ *makes this more apparent.*]

(b) We have

$$\begin{aligned}
&\frac{a_1}{\mu a_2 + \mu a_3 + \mu a_4} + \frac{a_2}{\lambda a_3 + \lambda a_4 + \lambda a_1} + \frac{a_3}{\mu a_4 + \lambda a_1 + \mu a_2} + \frac{a_4}{\lambda a_1 + \mu a_2 + \lambda a_3} \\
&\quad = \frac{a_1^2}{\mu a_2 a_1 + \mu a_3 a_1 + \mu a_4 a_1} + \frac{a_2^2}{\lambda a_3 a_2 + \lambda a_4 a_2 + \lambda a_1 a_2} \\
&\qquad + \frac{a_3^2}{\mu a_4 a_3 + \lambda a_1 a_3 + \mu a_2 a_3} + \frac{a_4^2}{\lambda a_1 a_4 + \mu a_2 a_4 + \lambda a_3 a_4} \\
&\geq \frac{(a_1 + a_2 + a_3 + a_4)^2}{(\lambda + \mu)(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)} \geq \frac{8}{3(\lambda + \mu)},
\end{aligned}$$

since $(a_1 + a_2 + a_3 + a_4)^2 \geq 8(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Almost all solvers showed that the inequalities hold for positive real numbers λ and μ.

3333. [2008 : 172, 174] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

The n points P_1, P_2, ..., P_n, labelled counterclockwise about a circle, form a convex n-gon Q. Denote by P_i' the point where the interior angle bisector at P_i intersects the circle. Suppose that the points P_i' determine another convex, cyclic n-gon Q', whose interior angle bisectors intersect the circle in the vertices of Q''. In this manner, we construct a sequence of convex, cyclic n-gons $Q^{(k)}$. For which values of n can we start with an n-gon that is not equiangular and arrive in k steps at an equiangular n-gon $Q^{(k)}$?

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove that n has the desired property if and only if it is a multiple of 4. Let R_0, R_1, ..., R_{n-1} be the vertices of $Q^{(k-1)}$, ϕ_0, ϕ_1, ..., ϕ_{n-1} be the corresponding interior angles, and S_0, S_1, ..., S_{n-1} be the vertices of $Q^{(k)}$. The subscripts are taken modulo n. Since the vertices are cyclic and labeled counterclockwise, for each j we have

$$\begin{aligned} \angle S_{j-1}S_jS_{j+1} &= \angle S_{j-1}S_jR_j + \angle R_jS_jS_{j+1} \\ &= \angle S_{j-1}R_{j-1}R_j + \angle R_jR_{j+1}S_{j+1} \\ &= \frac{\phi_{j-1}+\phi_{j+1}}{2}. \end{aligned} \tag{1}$$

Assume that $Q^{(k-1)}$ is not equiangular, whereas $Q^{(k)}$ is equiangular. The equality of the angles $\angle S_{j-1}S_jS_{j+1}$ in (1) implies $\phi_j+\phi_{j+2}=\phi_{j+2}+\phi_{j+4}$ for each j, that is,

$$\phi_j = \phi_{j+4}.$$

Thus, if n were odd, then all the ϕ_j would be equal, contradicting our assumption that $Q^{(k-1)}$ is not equiangular. Next, we suppose to the contrary that $n \equiv 2 \pmod 4$. We would then have $\phi_0=\phi_2=\cdots=\phi_{n-4}=\phi_{n-2}$, whence the circular arcs $R_{n-1}R_1$, R_1R_3, ..., $R_{n-3}R_{n-1}$ would be equal, and the points R_1, R_3, ..., R_{n-1} would be the vertices of a regular $\left(\frac{n}{2}\right)$-gon. It would then follow that $\phi_0=\phi_2=\cdots=\phi_{n-4}=\phi_{n-2}=\dfrac{(n-2)180^\circ}{n}$. Moreover, the same argument would apply to the odd subscripts; consequently, all the ϕ_j would be equal, again contradicting our hypothesis.

It remains to give an example for $Q^{(k-1)}$ with $n=4m$, where m is a positive integer: inscribe four regular $\left(\frac{n}{4}\right)$-gons in a circle in such a way that their initial vertices are not evenly spaced. For an explicit example with O the centre of the circumcircle, let

$$\angle R_{4j}OR_{4j+1} = \frac{135^\circ}{m}; \qquad \angle R_{4j+1}OR_{4j+2} = \frac{45^\circ}{m};$$
$$\angle R_{4j+2}OR_{4j+3} = \angle R_{4j+3}OR_{4j+4} = \frac{90^\circ}{m}.$$

Note that the interior angles, namely

$$\begin{aligned} \phi_{4j} &= 180^\circ - \frac{450^\circ}{n}\,, & \phi_{4j+1} &= 180^\circ - \frac{360^\circ}{n}\,, \\ \phi_{4j+2} &= 180^\circ - \frac{270^\circ}{n}\,, & \phi_{4j+3} &= 180^\circ - \frac{360^\circ}{n}\,, \end{aligned}$$

are not all equal. By (1) it is readily seen that $Q^{(k)}$ is equiangular.

Also solved by JOSEPH DiMURO and PETER Y. WOO, Biola University, La Mirada, CA, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incomplete submission.

DiMuro and Woo used a pictorial approach that focused on the short diagonals of the inscribed polygons—the chords joining R_{j-1} to R_{j+1}. An n-gon inscribed in a circle is equiangular if and only if these n chords have equal lengths. The point S_j where the bisector of $\angle R_{j-1}R_jR_{j+1}$ again meets the circle is where the perpendicular bisector of the chord $R_{j-1}R_{j+1}$ meets the corresponding arc that does not contain R_j. They prove that the derived n-gon Q' is necessarily convex whenever the initial n-gon Q is (so that the assumption of the convexity of Q' was not needed in the statement of the problem): the perpendicular bisector of the segment $R_{j-1}R_{j+1}$ passes through S_j, and one moves counterclockwise to reach the next chord R_jR_{j+2}, whose perpendicular bisector passes through S_{j+1}. By looking at the diagonals, one easily sees that if Q is any cyclic quadrilateral, then Q' is a rectangle (and is therefore equiangular); for an 8-gon, if the alternate vertices of $Q^{(k-1)}$ form two nonsquare rectangles, then it is not equiangular but $Q^{(k)}$, which consists of two squares, is equiangular. They raise the question of how wild the ancestors of $Q^{(k-1)}$ can be, but they do not pursue the answer.

3334. [2008 : 172, 175] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

(a) Prove that

$$\sum_{n=0}^{\infty} \frac{\sum\limits_{k=1}^{n+1} \frac{(-1)^{k-1}\binom{n+1}{k}}{k}}{(n+1)^2} \;=\; 2\zeta(3)\,.$$

(b) Prove that

$$\sum_{n=0}^{\infty} \frac{\sum\limits_{k=1}^{n+1} \frac{(-1)^{k-1}\binom{n+1}{k}}{k^2}}{(n+1)^2} \;=\; \frac{\pi^4}{30} \;=\; 3\zeta(4)\,.$$

[The function ζ is the Riemann Zeta Function: $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$.]

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $H_k = \sum\limits_{k=1}^{n} \frac{1}{k}$ be the n^{th} partial sum of the harmonic series. The following identity is well known [1]:

$$\sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k}\binom{n}{k} \;=\; H_n\,. \qquad (1)$$

The identity below has appeared in this journal [2]:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} \;=\; 2\zeta(3)\,. \tag{2}$$

From (1) and (2) we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^2}\sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k}\binom{n+1}{k}\right) \;=\; \sum_{n=0}^{\infty}\frac{H_{n+1}}{(n+1)^2} \;=\; 2\zeta(3)\,,$$

which establishes part (a).

To prove (b), we use the following identities due to D. Borwein and J.M. Borwein [3]:

$$\sum_{n=1}^{n}\frac{H_n^2}{(n+1)^2} = \frac{11}{4}\zeta(4)\,; \qquad \sum_{n=1}^{n}\frac{H_n}{n^3} = \frac{5}{4}\zeta(4)\,, \tag{3}$$

and the following summation trick [4]:

$$\sum_{1\le j<k\le n} a_j a_k \;=\; \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 - \sum_{k=1}^{n} a_k^2\right)\,. \tag{4}$$

Let $T_n = \sum\limits_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k^2}\binom{n+1}{k}$.

From (1) we have

$$\begin{aligned}
T_n &= \sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k^2}\left(\binom{n}{k}+\binom{n}{k-1}\right)\\
&= \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k^2}\binom{n}{k} \;+\; \sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k^2}\binom{n}{k-1}\\
&= T_{n-1} + \sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k(n+1)}\binom{n+1}{k} \;=\; T_{n-1} + \frac{H_{n+1}}{n+1}\,.
\end{aligned}$$

Since $T_0 = 1$, from this recurrence relation and induction we easily obtain

$$T_n \;=\; \sum_{k=1}^{n+1}\frac{H_k}{k}\,. \tag{5}$$

Using (4), we find that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{k^2}\right) &= \lim_{m\to\infty}\sum_{1\le k\le n\le m}\frac{1}{k^2n^2} \\
&= \lim_{m\to\infty}\left(\sum_{1\le k<n\le m}\frac{1}{k^2n^2}+\sum_{n=1}^{m}\frac{1}{n^4}\right) \\
&= \frac{1}{2}\left(\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2-\sum_{n=1}^{\infty}\frac{1}{n^4}\right) = \frac{1}{2}\left(\zeta^2(2)-\zeta(4)\right) \\
&= \frac{1}{2}\left(\frac{5}{2}\zeta(4)-\zeta(4)\right) = \frac{3}{4}\zeta(4)\,. \qquad (6)
\end{aligned}$$

[Ed: It is well known that $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$.]

Applying (3), (4), (5), and (6) we then have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^2}\sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k^2}\binom{n+1}{k}\right) &= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}T_n \\
= \sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^2}\sum_{k=1}^{n+1}\frac{H_k}{k}\right) &= \sum_{n=0}^{\infty}\frac{H_{n+1}}{(n+1)^3}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{(n+1)^2}\frac{H_k}{k}\right) \\
&= \frac{5}{4}\zeta(4) + \sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2}\sum_{1\le j\le k\le n}\frac{1}{jk}\right) \\
&= \frac{5}{4}\zeta(4) + \sum_{n=1}^{\infty}\left(\frac{1}{2(n+1)^2}\left(H_n^2+\sum_{k=1}^{n}\frac{1}{k^2}\right)\right) \\
&= \frac{5}{4}\zeta(4) + \frac{1}{2}\cdot\frac{11}{4}\zeta(4) + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{k^2}\right) \\
&= \left(\frac{5}{4}+\frac{11}{8}+\frac{3}{8}\right)\zeta(4) = 3\zeta(4) = \frac{\pi^4}{30}\,.
\end{aligned}$$

This completes the proof of part (b).

Also solved by MOHAMMED AASSILA, Strasbourg, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) *only); and the proposer.*

Aassila remarked that many similar and/or related identities can be found in the book Combinatorial Identities *by H.W. Gould.*

References

[1] Loren C. Larson, *Problem Solving Through Problems*, Springer-Verlag, New York, 1983, p. 160 [item 5.1.4.]

[2] ***Crux Mathematicorum with Mathematical Mayhem***, Solution to Problem 2984, Vol. 31, No. 7 (Sept., 2005) pp. 478-480.

[3] Jonathan Sondow and Eric W. Weisstein, MathWorld entry *Harmonic Number*, items 17 and 19, `http://mathworld.wolfram.com/HarmonicNumber.html`

[4] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, 1994, p. 37.

3335. [2008 : 172, 175] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let a and b be positive real numbers with $a < b$.

(a) Prove that

$$\frac{\ln b - \ln a}{b - a + 1} > \frac{b^{b-a} - a^{b-a}}{b^{b-a+1} - a^{b-a+1}}\,.$$

(b) Prove that

$$\int_a^b (x-a)^b (b-x)^a\,dx < \frac{1}{a+b+1}\left(b^{a+b+1} - a^{a+b+1}\right)\left(\frac{b-a}{b+a}\right)^{a+b}\,.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

Both inequalities are proved by using the General AM–GM Inequality, which states that if a_1, $a_2 > 0$ and w_1, $w_2 > 0$ with $w_1 + w_2 = 1$, then

$$a_1^{w_1} a_2^{w_2} \le w_1 a_1 + w_2 a_2\,,$$

equality holding if and only if $a_1 = a_2$.

(a) For $x \in (a,b)$ let $a_1 = a$, $a_2 = b$, $w_1 = \dfrac{b-x}{b-a}$, and $w_2 = \dfrac{x-a}{b-a}$. Since $0 < a < b$, the General AM–GM Inequality impies that

$$a^{\frac{b-x}{b-a}} b^{\frac{x-a}{b-a}} < \left(\frac{b-x}{b-a}\right) a + \left(\frac{x-a}{b-a}\right) b = x\,,$$

hence $a^{b-x} b^{x-a} < x^{b-a}$. It follows that

$$\int_a^b a^{b-x} b^{x-a}\,dx < \int_a^b x^{b-a}\,dx = \frac{b^{b-a+1} - a^{b-a+1}}{b-a+1}\,.$$

Since

$$\begin{aligned}
\int_a^b a^{b-x} b^{x-a}\,dx &= \frac{a^b}{b^a} \int_a^b \left(\frac{b}{a}\right)^x dx \\
&= \frac{a^b}{b^a} \left.\frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)}\right|_a^b = \frac{a^b}{b^a(\ln b - \ln a)}\left[\left(\frac{b}{a}\right)^b - \left(\frac{b}{a}\right)^a\right] \\
&= \frac{b^{b-a} - a^{b-a}}{\ln b - \ln a}\,,
\end{aligned}$$

we have
$$\frac{b^{b-a}-a^{b-a}}{\ln b-\ln a} \;<\; \frac{b^{b-a+1}-a^{b-a+1}}{b-a+1}\,,$$

and this is equivalent to the first inequality.

(b) For $x \in (a,b)$ let $a_1 = x-a$, $a_2 = b-x$, $w_1 = \dfrac{b}{a+b}$, and $w_2 = \dfrac{a}{a+b}$. The General AM–GM Inequality then implies

$$\begin{aligned}(x-a)^{\frac{b}{a+b}}(b-x)^{\frac{a}{a+b}} &\le \left(\frac{b}{a+b}\right)(x-a) + \left(\frac{a}{a+b}\right)(b-x)\\ &= \left(\frac{b-a}{b+a}\right)x\end{aligned}$$

with equality if and only if $x-a = b-x$, or if and only if $x = \dfrac{a+b}{2}$. Therefore, for all $x \in (a,b) - \left\{\frac{a+b}{2}\right\}$,

$$(x-b)^b(b-x)^a \;<\; \left(\frac{b-a}{b+a}\right)^{a+b} x^{a+b}\,,$$

and we have

$$\begin{aligned}\int_a^b (x-a)^b(b-x)^a\,dx &< \left(\frac{b-a}{b+a}\right)^{a+b}\int_a^b x^{a+b}\,dx\\ &= \frac{1}{a+b+1}\left(b^{a+b+1}-a^{a+b+1}\right)\left(\frac{b-a}{b+a}\right)^{a+b}.\end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAUL BRACKEN, University of Texas, Edinburg, TX, USA (in memory of James Totten); OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Bataille noted that the double integral $I = \int\int_\Delta \left(x^{b-a}-y^{b-a}\right)(1/x-1/y)\,dx\,dy$ is negative, where $\Delta = [a,b]\times[a,b]$, and he obtained the inequality upon expanding it. He remarked that if one considers $I = \int\int_\Delta\big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)\,dx\,dy$, where ϕ and ψ are continuous, monotonic real functions on $[a,b]$, then the method generalizes.

3336. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle, and let B_1 and B_2 be points on AC and C_1 and C_2 be points on AB such that $AB_1 = CB_2$, $AC_1 = BC_2$, and B_1C_2 intersects B_2C_1 at a point P in the interior of $\triangle ABC$. If $[KLM]$ denotes the area of $\triangle KLM$, show that

$$[PCB] \;>\; [PCA]+[PAB]\,.$$

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let M be the midpoint of AC and let N be the midpoint of AB; let $c' = AN(=\frac{1}{2}AB)$ and $b' = AM(=\frac{1}{2}AC)$. Assume that the points B_i and C_i lie on the sides AC and AB of $\triangle ABC$. Note that for P to lie inside the triangle, both B_1 and C_1 must be closer to A than B_2 and C_2 are, or both must be farther away. Let us label the points so that both are closer: B_1 lies between A and M while C_1 lies between A and N. Finally, let $m = MB_1 = MB_2$ and $n = NC_1 = NC_2$, and let NM meet B_1C_2 at Q and C_1B_2 at R. Our goal in what follows is to reduce the problem to proving that the points on NM are in the order N, Q, R, M.

Note that

$$[PCB] > [PCA] + [PAB] \text{ if and only if } [PCB] > \frac{1}{2}[ABC] \qquad (1)$$

(since $[ABC] = [PCB] + [PCA] + [PAB]$). The conditions in (1) hold if and only if P is inside $\triangle ANM$, which is true if and only if $\frac{NQ}{QM} < \frac{NR}{RM}$, or

$$\frac{NQ \cdot RM}{NR \cdot QM} < 1\,. \qquad (2)$$

By Menelaus' theorem applied to the transversal C_2QB_1 of triangle ANM,

$$\frac{NQ}{QM} = \frac{NC_2}{C_2A} \cdot \frac{AB_1}{B_1M} = \frac{n}{c'+n} \cdot \frac{b'-m}{m}\,.$$

Similarly, Menelaus' theorem applied to the transversal C_1RB_2 gives us

$$\frac{NR}{RM} = \frac{NC_1}{C_1A} \cdot \frac{AB_2}{B_2M} = \frac{n}{c'-n} \cdot \frac{b'+m}{m}\,.$$

Consequently,

$$\frac{NQ \cdot RM}{NR \cdot QM} = \frac{c'-n}{c'+n} \cdot \frac{b'-m}{b'+m}\,,$$

which is less than 1, as required by condition (2).

II. *Outline of the solution by Joel Schlosberg, Bayside, NY, USA, modified by the editor.*

Because ratios of areas, ratios of segments on a line or on parallel lines, and concurrence of lines are all affine properties, we may, without loss of generality, introduce a Cartesian coordinate system with

$$A = (0,1)\,, \quad B = (-2,-1)\,, \quad \text{and} \quad C = (2,-1)\,.$$

With these coordinates, the midpoints of AC and AB are, respectively,

$$M = (1,0) \quad \text{and} \quad N = (-1,0)\,.$$

As in the first solution, we take B_1 between A and M, and C_1 between A and N. Letting the vertical distance from M to B_1 be m and from N to C_1 be n, we have

$$\begin{aligned} B_1 &= (1-m, m)\,, & B_2 &= (1+m, -m)\,, \\ C_1 &= (-1+n, n)\,, & C_2 &= (-1-n, -n)\,. \end{aligned}$$

Again, as in the first solution, we wish to prove that the point P where B_1C_2 intersects C_1B_2 lies above the line NM, which here satisfies the equation $y = 0$. A straightforward computation shows that the y-coordinate of P is $mn > 0$, which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

In addition to his solution, Geupel showed that our statement of the problem was faulty: it should have explicitly stated that B_1 and B_2 are taken on the "side" AC (as opposed to the line AC), and similarly for the points C_1 and C_2 on side AB. Otherwise, with $m > 0$ and $n < 0$ in the notation of the second solution, it is easy to adjust these numbers so that P lies inside $\triangle ABC$ and below the line NM, in which case the inequality in the problem would be reversed. In fact, the proposer had it correct, but the word "side" was unfortunately omitted from the printed version.

3337. [2008 : 173, 175] *Proposed by Michel Bataille, Rouen, France.*

In the plane of $\triangle ABC$, what is the locus of points P such that the circumradii of $\triangle PBC$, $\triangle PCA$, and $\triangle PAB$ are all equal?

Solution by Oliver Geupel, Brühl, NRW, Germany.

Denote the desired locus by Π, and let Γ and H be the circumcircle and the orthocentre of $\triangle ABC$, respectively. We will prove that

$$\Pi = (\Gamma \cup \{H\}) \setminus \{A, B, C\}\,.$$

Let $P \in \Pi$. Then P is distinct from A, B, and C. [*Ed.: Whether or not one wants to include the vertices as part of the locus depends on how one interprets the question; most readers felt that the vertices should be omitted from the locus because a circumcircle is not well defined for a triangle when two of its vertices coincide.*] If two of the circumcircles of the triangles PBC, PCA, and PAB coincide, then all three circles coincide with Γ, and we conclude that $P \in \Gamma$. Otherwise, the centres of these congruent circumcircles will be distinct. Call them O_A, O_B, and O_C, respectively, and let A', B', and C' be the respective midpoints of the line segments PA, PB, and PC. Then $AC \parallel A'C'$. Moreover, O_C is the reflection of O_B in the mirror PA (which is the common chord of two congruent circles). Therefore, $O_BA' = A'O_C$; similarly, $O_BC' = C'O_A$, whence $A'C' \parallel O_CO_A$. We deduce that AC and

O_AO_C are parallel. Since PB is perpendicular to O_AO_C and, thus, to AC, the point P is on the altitude from B to CA. Similarly, P is on the other altitudes, whence $P = H$, as claimed.

Conversely, each point on Γ distinct from the vertices A, B, C belongs to the desired locus. It remains only to show that when ABC is not a right triangle (in which case H would be a vertex), the triangles HBC, HCA, and HAB have congruent circumcircles. This is a standard theorem—these circumcircles all have radii equal to the radius of Γ—but its proof is easy: Draw the lines through the vertices of $\triangle ABC$ that are parallel to the opposite sides, forming a triangle $A^*B^*C^*$. Let O_A, O_B, and O_C be the midpoints of HA^*, HB^*, and HC^*, respectively. Since A, B, and C are the midpoints of B^*C^*, C^*A^*, and A^*B^*, respectively, we see that H is the circumcentre of $\triangle A^*B^*C^*$, and

$$\begin{aligned} O_AB &= O_AH = O_AC = O_BC = O_BH \\ &= O_BA = O_CA = O_CH = O_CB\,. \end{aligned}$$

This completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There were five incomplete submissions.

Our result follows immediately from the 3*-circle Theorem of Gheorghe Ţiţeica from* 1908*:* If three congruent circles pass through a common point, then their other three intersection points lie on a fourth circle of the same radius. *The result was independently discovered by Roger A. Johnson ("A Circle Theorem",* Amer. Math. Monthly, **24***:5 (May, 1916), 243-244) and is sometimes attributed to him. Geupel's argument says that the four intersection points of these four congruent circles form an orthocentric quadrilateral (that is, each point is the orthocentre of the triangle formed by the other three). Because this configuration has been studied for such a long time, it seems likely that the result might be more than a century old.*