PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before August 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11474. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Valentin Vornicu, Aops-MathLinks forum, San Diego, CA.* (Correction) Show that when x, y, and z are greater than 1,

$$\Gamma(x)^{x^2+2yz}\Gamma(y)^{y^2+2zx}\Gamma(z)^{z^2+2xy} \geq (\Gamma(x)\Gamma(y)\Gamma(z))^{xy+yz+zx}.$$

11483. *Proposed by Éric Pité, Paris, France.* (Correction) The word "nonnegative" should read "positive."

11495. *Proposed by Marc Chamberland, Grinnell College, Grinnell, IA.* Let a, b, and c be rational numbers such that exactly one of $a^2b + b^2c + c^2a$, $ab^2 + bc^2 + ca^2$, and $a^3 + b^3 + c^3 + 6abc$ is zero. Show that $a + b + c = 0$.

11496. *Proposed by Benjamin Bogoşel, student, West University of Timisoara, Timisoara, Romania, and Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* For a matrix X with real entries, let $s(X)$ be the sum of its entries. Prove that if A and B are $n \times n$ real matrices, then

$$n\left(s(AA^T) + s(BB^T) - s(AB^T)s(A^TB)\right) \geq$$
$$s(AA^T)(s(B))^2 + s(BB^T)(s(A))^2 - s(A)s(B)\left(s(AB^T) + s(A^TB)\right).$$

11497. *Proposed by Mihály Bencze, Brasov, Romania.* Given n real numbers $x_1, \dots, x_n$ and a positive integer m, let $x_{n+1} = x_1$, and put

$$A = \sum_{k=1}^{n}\left(x_k^2 - x_kx_{k+1} + x_{k+1}^2\right)^m, \qquad B = 3\sum_{k=1}^{n} x_k^{2m}.$$

Show that $A \leq 3^m B$ and $A \leq (3^m B/n)^n$.

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11498. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Let $ABCD$ be a convex quadrilateral. A line through the intersection O of the diagonals AC and BD intersects the interior of edge BC at L and the interior of AD at N. Another line through O likewise meets AB at K and CD at M. This dissects $ABCD$ into eight triangles AKO, KBO, BLO, and so on. Prove that the arithmetic mean of the reciprocals of the areas of these triangles is greater than or equal to the sum of the arithmetic and quadratic means of the reciprocals of the areas of triangles ABO, BCO, CDO, and DAO. (The quadratic mean is also known as the root mean square; it is the square root of the mean of the squares of the given numbers.)

11499. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let H_n be the nth harmonic number, given by $H_n = \sum_{k=1}^{n} 1/k$. Let

$$S_k = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\log k - (H_{kn} - H_n)\right).$$

Prove that for $k \geq 2$,

$$S_k = \frac{k-1}{2k}\log 2 + \frac{1}{2}\log k - \frac{\pi}{2k^2}\sum_{l=1}^{\lfloor k/2 \rfloor}(k+1-2l)\cot\left(\frac{(2l-1)\pi}{2k}\right).$$

11500. *Proposed by Bhavana Deshpande, Poona College, Camp Pune, Maharashtra, India, and M. N. Deshpande, Institute of Science, Nagpur, India.* We have n balls, labeled 1 through n, and n urns, also labeled 1 through n. Ball 1 is put into a randomly chosen urn. Thereafter, as j increments from 2 to n, ball j is put into urn j if that urn is empty, otherwise, it is put into a randomly chosen empty urn. Let the random variable X be the number of balls that end up in the urn bearing their own number. Show that the expected value of X is $n - H_{n-1}$.

11501. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let

$$g(z) = 1 - \frac{3}{1 - \frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of g are all nonnegative if and only if $a \geq \sqrt{3}$.

SOLUTIONS

An Unusual GCD/LCM Relationship

11346 [2008, 167]. *Proposed by Christopher Hillar, Texas A&M University, College Station, TX, and Lionel Levine, University of California, Berkeley, CA.* Let n be an integer greater than 1, and let $S = \{2, \ldots, n\}$. For each nonempty subset A of S, let $\pi(A) = \prod_{j \in A} j$. Prove that when k is a positive integer and $k < n$,

$$\prod_{i=k}^{n}\operatorname{lcm}(\{1, \ldots, \lfloor n/i \rfloor\}) = \gcd(\{\pi(A)\colon\ |A| = n-k\}).$$

(In particular, setting $k = 1$ yields $\prod_{i=1}^{n}\operatorname{lcm}(\{1, \ldots, \lfloor n/i \rfloor\}) = n!$.)

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove that both sides equal $\prod_p p^{e_p(n,k)}$, where $e_p(n,k) = \sum_{i=k}^{n} \lfloor \log_p(n/i) \rfloor$ and the product runs over all primes (only finitely many primes contribute). Let $v_p(n)$ denote the maximum r such that p^r divides n.

For the left side, letting $l(x) = \operatorname{lcm}(\{1, \ldots, \lfloor x \rfloor\})$, we have $v_p(l(x)) = \lfloor \log_p x \rfloor$, since p^r divides $l(x)$ if and only if $x \geq p^r$. Hence $\prod_{i=k}^{n} l(n/i) = \prod_p p^{e_p(n,k)}$.

For the right side, let $(b_1, \ldots, b_{n-1})$ be the result of putting $(v_p(2), \ldots, v_p(n))$ in nonincreasing order. The number of terms with $v_p(k) \geq r$ equals the number of multiples of p^r in S, namely $\lfloor n/p^r \rfloor$. Thus $b_k \geq r$ if and only if $k \leq n/p^r$, and hence $b_k = \lfloor \log_p(n/k) \rfloor$. The smallest value of $v_p(\pi(A))$ such that $|A| = n-k$ will be achieved when A consists of exactly the elements of S corresponding to $b_k, \ldots, b_{n-1}$. Hence

$$v_p(\gcd(\{\pi(A)\colon \ |A| = n-k\})) = \sum_{i=k}^{n-1} b_i = e_p(n,k),$$

using the fact that the term for $i = n$ in the summation for $e_p(n,k)$ always equals 0. Applying this formula over all primes shows that the right side also equals $\prod_p p^{e_p(n,k)}$.

Also solved by D. R. Bridges, J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), T. Rucker, K. Schilling, A. Stadler (Switzerland), M. Tetiva (Romania), S. Vandervelde, B. Ward (Canada), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

Some Triangle Inequalities

11363 [2008, 461]. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* Let m_a, m_b, and m_c be the lengths of the medians of a triangle T. Similarly, let I_a, I_b, I_c, h_a, h_b, and h_c be the lengths of the bisectors and altitudes of T, and let R, r, and S be the circumradius, inradius, and area of T. Show that

$$\frac{I_a I_b}{I_c} + \frac{I_b I_c}{I_a} + \frac{I_c I_a}{I_b} \geq 3(2R - r),$$

and

$$\frac{m_a I_b}{h_c} + \frac{m_b I_c}{h_a} + \frac{m_c I_a}{h_b} \geq 3^{5/4}\sqrt{S}.$$

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. We write a, b, c for the lengths of the three sides, and $s = (a+b+c)/2$ for the semiperimeter. We will write $\sum$ or $\prod$ for a three or six term sum or product, respectively, over permutations of the triangle, with three terms if the sum is formally independent of the direction of the cycle, and six if not. Thus, $\sum ab$ denotes $ab + bc + ca$ while $\sum a^2 b = a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2$. We use several results from (or easily deduced from) *Geometric Inequalities* by Bottema et. al. (Nordhoff, Groningen, 1969), including:

$$I_a = \frac{2S}{(b+c)\sin(A/2)}, \quad abc = 4Rrs, \quad \frac{r}{4R} = \prod \sin\frac{A}{2},$$

$$\sum a^2 = 2(s^2 - 4Rr - r^2), \quad \sum a^2 b = 2s(s^2 - 2Rr + r^2),$$

$$\sum a^2 b^2 c = 4Rrs(s^2 + 4Rr + r^2),$$

$$\sum a^3 b^2 = 2s(s^4 + r^4 + 6Rr^3 + 8R^2 r^2 + 2r^2 s^2 - 10Rrs^2),$$

$$\sum a^4 b = 2s(s^4 - 3r^4 - 14Rr^3 - 8R^2 r^2 - 2r^2 s^2 - 6Rrs^2).$$

The first inequality must be reversed. In fact, we will show that

$$\frac{16}{9}(2R-r) < \frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} \le 3(2R-r).$$

We begin with

$$\sum \frac{I_aI_b}{I_c} = \sum \frac{\frac{2S}{(b+c)\sin(A/2)}\ \frac{2S}{(c+a)\sin(B/2)}}{\frac{2S}{(a+b)\sin(C/2)}}$$
$$= \frac{2S}{\prod(a+b)\prod\sin(A/2)}\sum(a+b)^2\sin^2\frac{C}{2}.$$

Now

$$2\sum(a+b)^2\sin^2\frac{C}{2} = \sum(a+b)^2(1-\cos C)$$
$$= 2\sum a^2 + 2\sum ab - \sum a^2\cos C - 2\sum ab\cos C.$$

But $2\sum ab\cos C = \sum(a^2+b^2-c^2) = \sum a^2$, so

$$2\sum a^2 + 2\sum ab - 2\sum ab\cos C = (\sum a)^2 = 4s^2$$

and

$$\sum a^2\cos C = \frac{1}{abc}\sum a^3bc\cos C = \frac{1}{2abc}\sum a^2c(a^2+b^2-c^2)$$
$$= \frac{1}{2abc}\left(\sum a^4c + 2\sum a^2b^2c - \sum a^2c^3\right)$$
$$= \frac{1}{4Rr}\big[s^4 - 3r^4 - 14Rr^3 - 8R^2r^2 - 2r^2s^2 - 6Rrs^2 + 4Rr(s^2 + 4Rr + r^2) - (s^4+r^4+6Rr^3+8R^2r^2+2r^2s^2-10Rrs^2)\big]$$
$$= \frac{2Rs^2 - 4Rr^2 - r^3 - rs^2}{R}.$$

Therefore

$$2\sum(a+b)^2\sin^2\frac{C}{2} = \frac{2Rs^2 + 4Rr^2 + r^3 + rs^2}{R}.$$

Furthermore, $\prod(a+b) = \sum a^2b + 2abc = 2s(s^2+2Rr+r^2)$ and $\prod\sin(A/2) = r/(4R)$. Hence

$$\frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} = \frac{2(2Rs^2+4Rr^2+r^3+rs^2)}{s^2+2Rr+r^2}. \qquad (*)$$

Now by *Geometric Inequalities* (5.9), $4R^2+4Rr+3r^2 \ge s^2 \ge r(16R-5r)$. For our lower bound: $2Rs^2 + 36Rr^2 + 17rs^2 + 17r^3 \ge 32R^2r + 26Rr^2 + 17rs^2 + 17r^3 > 32R^2r$, so $9(2Rs^2+4Rr^2+rs^2+r^3) > 8(2Rs^2+4R^2r-rs^2-r^3) = 8(s^2+2Rr+r^2)(2R-r)$. Hence

$$\frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} > \frac{16}{9}(2R-r).$$

For our upper bound: $R \geq 2r$, so $0 \leq (R-2r)(24R+10r)r = 24R^2r - 38Rr^2 - 10r^3$, and hence $44R^2r - 10Rr^2 \geq 20R^2r + 28Rr^2 + 20r^3$. Therefore $2Rs^2 + 12R^2r \geq 44R^2r - 10Rr^2 \geq 20R^2r + 28Rr^2 + 20r^3 \geq 8Rr^2 + 5rs^2 + 5r^3$, and $3(2R - r)(s^2 + 2Rr + r^2) = 6Rs^2 + 12R^2r - 3rs^2 - 3r^3 \geq 4Rs^2 + 8Rr^2 + 2r^3 + 2rs^2$. This inequality, in combination with $(*)$, gives

$$\frac{I_a I_b}{I_c} + \frac{I_b I_c}{I_a} + \frac{I_c I_a}{I_b} \leq 3(2R - r).$$

Now consider the second inequality. By elementary calculus, a function of the form $f(x) = x^2 + 2\lambda/x$ achieves its minimum at $x = \lambda^{1/3}$, so $f(x) \geq 3\lambda^{2/3}$.

Letting $\lambda = \prod m_a I_b / h_c$, we have

$$\left(\sum \frac{m_a I_b}{h_c}\right)^2 = \sum \frac{m_a^2 I_b^2}{h_c^2} + 2\sum \frac{m_a I_b}{h_c}\frac{m_b I_c}{h_a} = \sum \left(\frac{m_a^2 I_b^2}{h_c^2} + 2\lambda \frac{h_c}{m_a I_b}\right) \geq 9\lambda^{2/3}.$$

Denote the exradii of T by r_a, r_b, and r_c. By *Geometric Inequalities* (8.21) and (6.27), we have $m_a m_b m_c \geq r_a r_b r_c = S^2/r = Ss$. By (8.7) we have

$$\begin{aligned} I_a I_b I_c &= \frac{8a^2b^2c^2}{\prod(a+b)} \prod \cos\frac{A}{2} = \frac{8a^2b^2c^2}{\prod(a+b)} \prod \sqrt{\frac{s(s-a)}{bc}} \\ &= \frac{8a^2b^2c^2}{\prod(a+b)} \frac{Ss}{abc} = \frac{8abcSs}{\prod(a+b)} = \frac{32RsS^2}{\prod(a+b)}, \\ h_a h_b h_c &= \prod \frac{2S}{a} = \frac{8S^3}{abc} = \frac{2S^3}{Rrs}. \end{aligned}$$

Now

$$\lambda = Ss \frac{32RsS^2}{\prod(a+b)} \frac{Rrs}{2S^3} = \frac{16R^2rs^3}{\prod(a+b)} \quad \text{and} \quad \left(\sum \frac{m_a I_b}{h_c}\right)^2 \geq 9\left(\frac{16R^2rs^3}{\prod(a+b)}\right)^{2/3}.$$

By (5.5) and (5.1), $s^2 \geq 3r(4R + r) \geq 3r(9r) = 27r^3$, so $s \geq 3\sqrt{3}r$. By (5.8) $s^2 \leq 4R^2 + 4Rr + 3r^2$, and thus $s^2 + 2Rr + r^2 \leq 4R^2 + 6Rr + 4r^2 \leq 4R^2 + 3R^2 + R^2 = 8R^2$. Hence $\prod(a+b) = \sum a^2b + 2abc = 2s(s^2 - 2Rr + r^2) + 8Rrs = 2s(s^2 + 2Rr + r^2) \leq 2s(8R^2) = 16R^2s$. This leads to $3\sqrt{3}(\prod(a+b))^2 \leq s(16R^2s)^2 = 256R^4s^3$. Now $3^{15/2}S^3 = 3^{15/2}r^2s^3$, and

$$3^{15/2}r^3s^3 \leq 729\frac{256R^4r^2s^6}{\left(\prod(a+b)\right)^2} \Rightarrow 3^{5/2}S \leq 9\left(\frac{16R^2rs^3}{\prod(a+b)}\right)^{2/3} \leq \left(\sum \frac{m_a I_b}{h_c}\right)^2,$$

so that finally $3^{5/4}\sqrt{S} \leq \sum m_a I_b / h_c$.

Also solved by V. V. García (Spain) and R. Stong.

A Multiple of a Prime

11364 [208, 461]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let p be a prime greater than 3, and let t be the integer nearest $p/6$.

(**a**) Show that if $p = 6t + 1$, then

$$(p-1)! \sum_{j=0}^{2t-1} (-1)^j \left(\frac{1}{3j+1} + \frac{1}{3j+2}\right) \equiv 0 \pmod{p}.$$

(**b**) Show that if $p = 6t - 1$, then

$$(p-1)!\left(\sum_{j=0}^{2t-1} \frac{(-1)^j}{3j+1} + \sum_{j=0}^{2t-2} \frac{(-1)^j}{3j+2}\right) \equiv 0 \pmod{p}.$$

Solution by Robin Chapman, University of Exeter, Exeter, U. K. The desired congruence in both cases is

$$(p-1)! \sum_{k=1}^{p-1} \frac{\chi(k)}{k} \equiv 0 \pmod{p}, \tag{1}$$

where

$$\chi(k) = \begin{cases} 0 & \text{if } k \equiv 0, 3 \pmod{6}, \\ 1 & \text{if } k \equiv 1, 2 \pmod{6}, \\ -1 & \text{if } k \equiv 4, 5 \pmod{6}. \end{cases}$$

Note that $\chi(k) = (\zeta^k - \zeta^{-k})/\sqrt{-3}$, where $\zeta = e^{\pi i/3} = \frac{1}{2}(1+\sqrt{-3})$. Letting $F(z) = \sum_{k=1}^{p-1} z^k/k$, we have

$$\sum_{k=1}^{p-1} \frac{\chi(k)}{k} = \frac{F(\zeta) - F(\zeta^{-1})}{\sqrt{-3}}. \tag{2}$$

For the value on the right, note that $F'(z) = \sum_{k=1}^{p-1} z^{k-1} = \frac{1-z^{p-1}}{1-z}$, so $F'(1-z) = \sum_{k=0}^{p-2}(-1)^{k+1}\binom{p-1}{k+1}z^k$. Note also that $\binom{p-1}{j} \equiv (-1)^j \pmod{p}$. Hence $F'(1-z) = pG(z) + F'(z) \pmod{p}$, where G is a polynomial having integer coefficients and degree at most $p-2$. We conclude that

$$\frac{d}{dz}(F(z) - F(1-z)) = -pG(z). \tag{3}$$

Let $G(z) = \sum_{k=1}^{p-1} b_k z^{k-1}$ with each $b_k \in \mathbb{Z}$. Integrating (3) from 0 to z gives

$$F(z) - F(1-z) + F(1) = -p \sum_{k=1}^{p-1} \frac{b_k}{k} z^k.$$

Setting $z = \zeta$ and using $1 - \zeta = \zeta^{-1}$ yields

$$F(\zeta) - F(\zeta^{-1}) = -F(1) - p \sum_{k=1}^{p-1} \frac{b_k}{k} z^k.$$

Since p is odd, $F(1) = \sum_{k=1}^{(p-1)/2}\left(\frac{1}{k} + \frac{1}{p-k}\right) = \sum_{k=1}^{(p-1)/2} \frac{p}{k(p-k)}$. It follows that $(p-1)!\,F(1)$ is a multiple of p. We conclude that in the context of algebraic integers, $(p-1)!\,(F(\zeta) - F(\zeta^{-1})) \equiv 0 \pmod{p}$. Multiplying by $\sqrt{-3}$ yields a rational integer, and dividing by -3 (justified by $p > 3$) and invoking (2) yields the desired congruence (1).

Editorial comment. Stong showed also that $(p-1)!\,F(\zeta) \equiv (p-1)!\,F(\zeta^{-1}) \equiv 0 \pmod{p}$, which leads to $(p-1)! \sum_{k=1}^{p-1} \frac{\chi(k+s)}{k} \equiv 0 \pmod{p}$ for every integer s.

Also solved by J. H. Lindsey II, M. A. Prasad (India), A. Stadler (Switzerland), R. Tauraso (Italy), M. Tetiva (Romania), A. Wyn-Jones, GCHQ Problem Solving Group (U. K.), and the proposer.

Relating Two Integer Sequences

11365 [2008, 462]. *Proposed by Aviezri S. Fraenkel, Weizmann Institute of Science, Rehovot, Israel.* Let t be a positive integer. Let $\gamma = \sqrt{t^2+4}$, $\alpha = \frac{1}{2}(2+\gamma-t)$, and $\beta = \frac{1}{2}(2+\gamma+t)$. Show that for all positive integers n,

$$\lfloor n\beta \rfloor = \lfloor (\lfloor n\alpha \rfloor + n(t-1))\alpha \rfloor + 1 = \lfloor (\lfloor n\alpha \rfloor + n(t-1)+1)\alpha \rfloor - 1.$$

Solution I by Donald R. Bridges, Woodstock, MD. Letting $\epsilon = (\gamma - t)/2$, we have $\alpha = 1+\epsilon$ and $\beta = 1+t+\epsilon$. Note that $t^2 < \gamma^2 < (t+2)^2$, so γ and ϵ are irrational and $0 < \epsilon < 1$.

We write the expressions in terms of ϵ. For the first, $\lfloor n\beta \rfloor = n + nt + \lfloor n\epsilon \rfloor$. For the second,

$$\lfloor n\alpha \rfloor + n(t-1) = nt + \lfloor n\epsilon \rfloor,$$
$$(\lfloor n\alpha \rfloor + n(t-1))\alpha = nt + \lfloor n\epsilon \rfloor + nt\epsilon + \lfloor n\epsilon \rfloor \epsilon.$$

Squaring both sides of $\sqrt{t^2+4} = t + 2\epsilon$ yields $t\epsilon + \epsilon^2 = 1$, so $nt\epsilon + n\epsilon^2 = n$. Also, $nt\epsilon + \lfloor n\epsilon \rfloor \epsilon > nt\epsilon + (n\epsilon - 1)\epsilon$, so the floor of the last displayed expression is $nt + \lfloor n\epsilon \rfloor + n - 1$, since $0 < \epsilon < 1$. This proves the first equality.

To compute the rightmost expression in the problem statement, begin with

$$(\lfloor n\alpha \rfloor + n(t-1)+1)\alpha = nt + \lfloor n\epsilon \rfloor + 1 + nt\epsilon + \lfloor n\epsilon + 1 \rfloor \epsilon.$$

Since $nt\epsilon + \lfloor n\epsilon + 1 \rfloor \epsilon \le nt\epsilon + n\epsilon^2 + \epsilon < n+1$, we obtain the desired equality

$$\lfloor (\lfloor n\alpha \rfloor + n(t-1)+1)\alpha \rfloor = \lfloor n\beta \rfloor + 1.$$

Solution II by the proposer. First, observe that α and β are irrational numbers satisfying $1 < \alpha < \beta$ and $\alpha + \beta = \alpha\beta$, and that as a result, $\beta > 2$. It is well known that under these conditions, $A \cup B = \mathbb{N}$, where $A = \{\lfloor n\alpha \rfloor : n \ge 1\}$ and $B = \{\lfloor n\beta \rfloor : n \ge 1\}$.

Since $\beta > 2$, the set B does not contain consecutive integers. Hence each term of B lies between two consecutive terms of A. That is, for each positive integer n there exists m such that $\lfloor m\alpha \rfloor$, $\lfloor n\beta \rfloor$, and $\lfloor (m+1)\alpha \rfloor$ are consecutive integers. Given n, the problem is to determine m.

Among the integers from 1 to $\lfloor n\beta \rfloor$, exactly n lie in B, so $\lfloor n\beta \rfloor - n$ lie in A. Therefore, $m = \lfloor n\beta \rfloor - n$. Thus

$$\lfloor (\lfloor n\beta \rfloor - n)\,\alpha \rfloor, \qquad \lfloor n\beta \rfloor, \qquad \lfloor ((\lfloor n\beta \rfloor - n) + 1)\,\alpha \rfloor$$

are consecutive integers. It remains only to show that $\lfloor n\beta \rfloor - n = \lfloor n\alpha \rfloor + n(t-1)$. This reduces to $\left\lfloor \frac{1}{2}n(\gamma+t) \right\rfloor = \left\lfloor \frac{1}{2}n(\gamma-t) \right\rfloor + nt$, which is true.

Editorial comment. The claim that $A \cup B = \mathbb{N}$ in Solution II is well known; the proposer cited A. S. Fraenkel, How to beat your Wythoff games opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353–361. The result is so astonishing and yet easily proved that we include a short proof for the reader's pleasure.

First note that $a + b = ab$ is equivalent to $\frac{1}{a} + \frac{1}{b} = 1$. Also, $a, b > 1$. For any $k \in \mathbb{N}$, the number of terms less than k in $A \cup B$ is $\lfloor k/a \rfloor + \lfloor k/b \rfloor$, since a and b are irrational. We compute

$$\left\lfloor \frac{k}{a} \right\rfloor + \left\lfloor \frac{k}{b} \right\rfloor = \left\lfloor \frac{k}{a} \right\rfloor + \left\lfloor k\left(1 - \frac{1}{a}\right) \right\rfloor = k + \left\lfloor \frac{k}{a} \right\rfloor + \left\lfloor \frac{-k}{a} \right\rfloor = k - 1.$$

Similarly, $A \cup B$ contains k terms less than $k+1$. Hence there is exactly one term less than $k+1$ but not less than k; it equals k.

Also solved by R. Chapman (U. K.), P. Corn, C. Curtis, J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

An Exponential Inequality

11369 [2008, 567]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that for all real t, and all $\alpha \geq 2$,

$$e^{\alpha t} + e^{-\alpha t} - 2 \leq \left(e^t + e^{-t}\right)^\alpha - 2^\alpha.$$

Solution by Knut Dale, Telemark University College, Bø, Norway. For $t \in \mathbb{R}$ and $\alpha \geq 0$, let $f(t,\alpha) = ((e^t + e^{-t})^\alpha - 2^\alpha) - (e^{\alpha t} + e^{-\alpha t} - 2)$. Since $f(0,\alpha) = 0$ and $f(-t,\alpha) = f(t,\alpha)$, we need only consider $t > 0$. Write

$$\begin{aligned} f(t,\alpha) &= \alpha \int_0^t \left\{ (e^x + e^{-x})^\alpha \frac{\sinh x}{\cosh x} - (e^{\alpha x} - e^{-\alpha x}) \right\} dx \\ &= \alpha \int_0^t (e^x + e^{-x})^\alpha \left\{ g(x,1) - g(x,\alpha) \right\} dx, \end{aligned}$$

where $g(x,\alpha) = (e^{\alpha x} - e^{-\alpha x})/(e^x + e^{-x})^\alpha$. Let $x > 0$ and observe that $g(x,\alpha) \geq 0$, $g(x,2) = g(x,1) > 0$, and $g(x,0) = g(x,\infty) = 0$. Note that

$$\frac{\partial g(x,\alpha)}{\partial \alpha} > 0 \quad \Longleftrightarrow \quad \frac{\ln(e^x + e^{-x}) + x}{\ln(e^x + e^{-x}) - x} > e^{2\alpha x}. \qquad (*)$$

Likewise, equivalence holds if we replace "$>$" with "$=$" or with "$<$" throughout $(*)$. Since $e^{2\alpha x}$ is an increasing function of α,

$$\frac{\ln(e^x + e^{-x}) + x}{\ln(e^x + e^{-x}) - x} = e^{2\alpha x}$$

has a unique solution α in the interval $(1,2)$. Thus, as a function of α, $g(x,\alpha)$ increases from 0 to a maximum in $(1,2)$ and then decreases towards 0. Hence $f(t,\alpha) > 0$ for $\alpha \in (0,1) \cup (2,\infty)$, $f(t,\alpha) < 0$ for $\alpha \in (1,2)$, and $f(t,\alpha) = 0$ for $\alpha \in \{0,1,2\}$.

Editorial comment. Grahame Bennett (Indiana University) provided an instructive solution including a general context for this inequality. That solution is now incorporated into a paper, appearing in the current issue of this MONTHLY (see p. 334).

Also solved by F. Alayont, K. Andersen (Canada), R. Bagby, G. Bennett, D. & J. Borwein (Canada), P. Bourdon, P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), K. Endo, G. C. Greubel, J. Grivaux (France), J. A. Grzesik, S. J. Herschkorn, M. Hildebrand, F. Holland (Ireland), A. Incognito & T. Mengesha, V. K. Jenner (Switzerland), O. Kouba (Syria), K.-W Lau (China), W. R. Livingston, O. P. Lossers (Netherlands), K. McInturff, K. Nagasaki (Japan), T. Nakata (Japan), O. Padé (Israel), P. Perfetti (Italy), Á. Plaza & J. M. Pacheco (Spain), D. S. Ross, V. Rutherfoord, B. Schmuland (Canada), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), M. Thaler (Australia), J. Vinuesa (Spain), Z. Vörös (Hungary), T. Wilkerson, Y. Yu, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.